

ENTROPY, STABILITY, AND YANG-MILLS FLOW

CASEY KELLEHER AND JEFFREY STREETS

ABSTRACT. Following [3], we define a notion of entropy for connections over \mathbb{R}^n which has shrinking Yang-Mills solitons as critical points. As in [3], this entropy is defined implicitly, making it difficult to work with analytically. We prove a theorem characterizing entropy stability in terms of the spectrum of a certain linear operator associated to the soliton. This leads furthermore to a gap theorem for solitons. These results point to a broader strategy of studying “generic singularities” of Yang-Mills flow, and we discuss the differences in this strategy in dimension $n = 4$ versus $n \geq 5$.

1. INTRODUCTION

In [3] Colding and Minicozzi introduced a strategy for understanding the mean curvature flow based on a notion of entropy-stability of singularities. Broadly speaking, the goal is to showing that all singularities other than cylinders and spheres are unstable and hence can be perturbed away, leading to the construction of “generic” mean curvature flows. In this paper we initiate a similar strategy for understanding the Yang-Mills flow. We first recall some basic setup and fundamental results about this flow. In [13] Rade showed the smooth long time existence and convergence of Yang-Mills flow in dimensions $n = 2, 3$. Next, in [16], Struwe gave a criterion for singularity formation in dimension $n = 4$ in terms of energy concentration, although it is to date an open problem whether or not such energy concentration occurs. Explicit finite time singularities of Yang-Mills flow in dimensions $5 \leq n \leq 9$ were constructed in [4]. Following [3], we seek to investigate what a “stable,” or “generic,” singularity of Yang-Mills flow looks like. Similar constructions for the harmonic map flow were considered in [22].

In [7], Hamilton defined an entropy functional for Yang-Mills flow akin to Huisken’s monotonicity formula for mean curvature flow [8], and Struwe’s monotonicity for harmonic maps [17], which is monotone against special background manifolds (with an easily controlled decay rate in general). In [20] Weinkove used this monotonicity formula to establish that type I singularities of Yang-Mills flow admit blowup limits which are shrinking soliton solutions of Yang-Mills flow (Definition 2.13). At this point, one may ask whether or not *all* singularities of Yang-Mills flow admit shrinking soliton blowup limits. The corresponding statement for mean curvature flow was established in work of Ilmanen, White [9, 21]. Because of this it is reasonable to initiate an in-depth study of mean curvature flow shrinkers to define the notion of a stable singularity. Despite the fact that it is unknown if all Yang-Mills flow singularities in dimension $n \geq 5$ can be described by shrinkers, we will nonetheless use them as our models to define a notion of stable singularity.

In particular, we draw inspiration from Colding-Minicozzi’s approach to Huisken’s monotonicity formula directly and explicitly include a base point in the definition of Hamilton’s Yang-Mills

Date: October 17, 2014.

The first author was supported by an NSF Graduate Research Fellowship DGE-1321846. The second author was partly supported by the National Science Foundation DMS-1341836 and an Alfred P. Sloan Fellowship.

entropy. In particular, we will set

$$\mathcal{F}_{x_0, t_0}(\nabla) = t_0^2 (4\pi t_0)^{-\frac{n}{2}} \int_{\mathbb{R}^n} |F_\nabla|^2 e^{-\frac{|x-x_0|^2}{4t_0}} dV$$

This functionals have the key property that their critical points are self-shrinking solutions to Yang-Mills flow. Moreover, the the entropy takes the supremum of \mathcal{F} over points in spacetime, specifically

$$\lambda(\nabla) = \sup_{x_0, t_0} \mathcal{F}_{x_0, t_0}(\nabla).$$

This quantity has many invariance properties, and is moreover monotone along a solution to Yang-Mills flow (Proposition 3.8). However, it does not depend smoothly on ∇ , and thus is difficult to work with analytically. Nonetheless, we are able to relate stability with respect to ∇ to the more computationally tractable stability coming from the \mathcal{F} -functionals (see Definition 4.1). This stability comes from a more traditional second variation analysis for the functional.

In analyzing the \mathcal{F} -functionals, in parallel to [3], we observe that two negative eigenforms for this second variation are always present, corresponding to the Yang-Mills flow direction and also translation in space. Taking these explicitly into account yields an analytic characterization of the formal definition of the \mathcal{F} -stability of a shrinking soliton (see Definition 4.1, Theorem 4.5). Also, as a consequence of this analysis of the second variation, we establish a basic gap theorem for solitons.

Theorem 1.1. *A soliton satisfying $|F_\nabla| \leq \frac{3}{8}$ is flat.*

Putting this discussion together and exploiting a number of interesting properties of the \mathcal{F} -functionals, we are able to relate \mathcal{F} stability with entropy stability. This theorem is analogous to ([3] Theorem 0.15).

Theorem 1.2. *Suppose $\nabla \in \mathfrak{S}$ is non cylindrical with polynomial curvature growth. If ∇ is \mathcal{F} -unstable then there is a compactly supported variation ∇_s such that $\nabla_0 = \nabla$ and for all $s \neq 0$,*

$$(1.1) \quad \lambda(\nabla_s) < \lambda(\nabla).$$

We note here that the only known examples of type I blowups ([5, 12]) and shrinking solitons ([4]) are in dimensions $n \geq 5$, whose entropies we compute in §6. We will also show that this dimensional restriction is necessary. In particular, while type I blowup limits are shrinking solitons, we show that any shrinking soliton on \mathbb{R}^4 is automatically flat, thus ruling out type I blowups.

Proposition 1.3. *Let $E \rightarrow (M^4, g)$ be a smooth vector bundle, and suppose ∇_t is a solution to Yang-Mills flow on E . If $T < \infty$ is the maximal existence time of the solution, then*

$$\lim_{t \rightarrow T} (T - t) |F_{\nabla_t}| = \infty.$$

Moreover, any shrinking soliton on \mathbb{R}^n , $n \leq 4$, is flat.

For this reason in four-dimensions it is more natural to define stability using the second variation of the Yang-Mills functional itself, not the entropy. This notion was introduced in [1], where the second variation of \mathcal{YM} is analyzed in depth. Among many results in that paper is a strong rigidity result showing that for $SU(2)$, $SU(3)$ or $SO(4)$ -bundles over S^4 , the only stable Yang-Mills connections are either self-dual or antiself-dual. However, a recent result of Waldron [19] suggests that blowup limits of finite time singularities of Yang-Mills flow in dimension $n = 4$ are *not* instantons, and hence should not be stable. This suggests that it may be possible to construct smooth long time Yang-Mills flows “generically” for vector bundles on four-manifolds with small gauge group.

Given the results above, a number of natural questions emerge following parallel lines of thought from mean curvature flow. First, we note that for solutions to mean curvature flow it was shown in [9, 21] that arbitrary singularities admits blowup limits which are shrinking solitons. A very natural question is whether the result of [20] can be extended to show the analogous statement for Yang-Mills flow. In other words, do arbitrary singularities of Yang-Mills flow admit shrinking soliton blowup limits? Another basic issue is to construct more examples of shrinking solitons. Despite the lack of examples, one would still like to know the answers to some simple questions. For instance, for a given gauge group, what is the minimal entropy shrinker? Also, is it possible to classify stable shrinkers?

Here is an outline of the rest of this paper. In §2 we recall Hamilton's general entropy functional and monotonicity formula for Yang-Mills flow, as well as a specialized version of this functional on \mathbb{R}^n adapting Huisken's entropy for mean curvature flow to Yang-Mills flow. In §3 we establish variational properties of the entropy, leading to the some corollaries on the structure of self-shrinkers of Yang-Mills flow. Next, in §4 we make some observations about the spectrum of the second variation of \mathcal{F} , leading to a characterization of \mathcal{F} -stability and the proof of Theorem 1.1. In §5 we prove Theorem 1.2, and we conclude in §6 by computing the entropy of the Gastel shrinkers.

Acknowledgements: The authors would like to thank Michael Struwe for his comments on an earlier draft of this paper.

2. \mathcal{F} -FUNCTIONAL AND ENTROPY

2.1. Background. Let (M^n, g) be a smooth, compact Riemannian manifold without boundary. Given a vector bundle E over M , let $S(E)$ denote the sections of E . For each point p choose a local basis of TM given by $\{\partial_i\}$ with dual elements $\{dx^i\}$, where $dx^i(\partial_j) = \delta_j^i$. Additionally, choose a local basis for E given by $\{\mu_\alpha\}$ with dual elements $\{\mu_\alpha^*\}$ where $\mu_\alpha^*(\mu_\beta) = \delta_\alpha^\beta$. Given a chart containing $p \in M$ the action of a connection ∇ on E is given in a local basis by

$$\nabla \mu_\beta = \Gamma_{i\beta}^\delta dx^i \otimes \mu_\delta.$$

Set $\Gamma = (\Gamma_{i\beta}^\delta dx^i \otimes \mu_\delta \otimes \mu_\beta^*)$ to be the *connection coefficient matrix* (associated with ∇) with respect to the basis. The set of all connections over M will be denoted by \mathcal{A}_E . The actions of ∇ are extended to TM by coupling it with the unique Levi-Civita connection of (M, g) , given locally via

$$\nabla \partial_j = (\Gamma^{LC})_{ij}^k dx^i \otimes \partial_k.$$

The actions of ∇ may be extended to tensorial combinations of T^*M and E as well as their dual spaces. We let ∇^* denote the formal adjoint of ∇ with respect to the inner product.

Let D be the *exterior derivative*, or skew symmetrization of ∇ over the tensor products of T^*M . Set $\Lambda^p(E) = \Lambda^p(M) \otimes S(E)$. We let $D^{(p)}$ be the covariant connection from $\Lambda^p(E)$ to $\Lambda^{p+1}(E)$, where the p index will be dropped when understood. The curvature tensor $F_\nabla := D^{(1)} \circ D^{(0)} : \Lambda^0(E) \rightarrow \Lambda^2(E)$ is given in local coordinates by

$$(2.1) \quad F_\nabla = \left(\partial_i \Gamma_{j\alpha}^\beta - \partial_j \Gamma_{i\alpha}^\beta - \Gamma_{i\alpha}^\delta \Gamma_{j\delta}^\beta + \Gamma_{j\alpha}^\delta \Gamma_{i\delta}^\beta \right) dx^i \wedge dx^j \otimes \mu_\beta \otimes \mu_\alpha^*.$$

Three more operators will be particularly important to our study. We set $D_\nabla^* := \nabla^*$, which is a rescaled version of the formal L^2 adjoint of D , chosen for computational convenience. The *Hodge Laplacian* is given by

$$\Delta_{D_\nabla} : \Lambda^p(E) \rightarrow \Lambda^p(E) : \omega \mapsto (D_\nabla^* D_\nabla + D_\nabla D_\nabla^*) \omega,$$

while the *rough (Bochner) Laplacian* and is given by

$$\Delta : \Lambda^p(E) \rightarrow \Lambda^p(E) : \omega \mapsto -\nabla^* \nabla \omega.$$

We next discuss background material pertinent to the study of the Yang-Mills functional and its generalizations. We define the *Yang-Mills functional* by

$$(2.2) \quad \mathcal{YM}(\nabla) := \|F_\nabla\|_{L^2}^2 = \int_M |F_\nabla|^2 dV_g.$$

By computing the Euler Lagrange equation of (2.2) one may generate the corresponding *Yang-Mills flow* defined as follows.

$$(2.3) \quad \frac{\partial \nabla_t}{\partial t} = -D_{\nabla_t}^* F_{\nabla_t}.$$

Let J and K be multiindices of lengths $p_J - 1$ and $p_K - 1$ respectively, where $J := (j_i)_{i=1}^{p_J-1}$ and $K := (k_i)_{i=1}^{p_K-1}$. The operation *pound* is given by

$$\begin{aligned} \# : (T^*M)^{\otimes p_J} \otimes \text{End}(E) \times (T^*M)^{\otimes p_K} \otimes \text{End}(E) &\rightarrow (T^*M)^{p_J+p_K-2} \otimes \text{End}(E) : (A, B) \mapsto A \# B, \\ (A \# B)(\partial_{j_1}, \dots, \partial_{j_{p_J-1}}, \partial_{k_1}, \dots, \partial_{k_{p_K-1}}) &:= \sum_{i=1}^n A(\partial_i, \partial_{j_1}, \dots, \partial_{j_{p_J-1}}) B(\partial_i, \partial_{k_1}, \dots, \partial_{k_{p_K-1}}). \end{aligned}$$

In coordinates this is written in the form $(A \# B)_{JK\alpha}^\beta = g^{jk} A_{jJ\delta}^\beta B_{kK\alpha}^\delta$. Roughly speaking, $\#$ is matrix multiplication combined with contraction of the first two forms.

Lastly we define the *pound bracket* by

$$\begin{aligned} [\cdot, \cdot]^\# : (T^*M)^{\otimes p_J} \otimes \text{End}(E) \times (T^*M)^{\otimes p_K} \otimes \text{End}(E) &\rightarrow (T^*M)^{p_J+p_K-2} \otimes \text{End}(E) \\ : (A, B) &\mapsto A \# B - B \# A. \end{aligned}$$

Lemma 2.1. *Given ∇ a connection and $\omega \in \Lambda^2(\text{End } E)$, one has*

$$(2.4) \quad D^* D^* \omega = \frac{g^{il} g^{jk}}{2} \left(F_{ij\delta}^\beta \omega_{kl\alpha}^\delta - F_{ij\alpha}^\delta \omega_{kl\delta}^\beta \right),$$

and in particular $D^* D^* F = 0$.

Proof. We compute

$$\begin{aligned} g^{il} g^{jk} \nabla_i \nabla_j \omega_{kl\alpha}^\beta &= \frac{g^{il} g^{jk}}{2} \left(\nabla_i \nabla_j \omega_{kl\alpha}^\beta - \nabla_i \nabla_j \omega_{lk\alpha}^\beta \right) \\ &= \frac{g^{il} g^{jk}}{2} [\nabla_i, \nabla_j] \omega_{kl\alpha}^\beta \\ &= \frac{g^{il} g^{jk}}{2} \left(\text{Rm}_{ijk}^p \omega_{p\ell\alpha}^\beta + \text{Rm}_{ij\ell}^p \omega_{kp\alpha}^\beta - F_{ij\alpha}^\delta \omega_{k\ell\delta}^\beta + F_{ij\delta}^\beta \omega_{k\ell\alpha}^\delta \right) \\ &= \frac{g^{il} g^{jk}}{2} \left(g^{pq} \left(\text{Rm}_{jilq} \omega_{pk\alpha}^\beta + \text{Rm}_{ijlq} \omega_{kp\alpha}^\beta \right) - F_{ij\alpha}^\delta \omega_{k\ell\delta}^\beta + F_{ij\delta}^\beta \omega_{k\ell\alpha}^\delta \right) \\ &= g^{il} g^{jk} \left(g^{pq} \left(\text{Rm}_{ijlq} \omega_{kp\alpha}^\beta \right) - \frac{1}{2} \left(F_{ij\alpha}^\delta \omega_{k\ell\delta}^\beta + F_{ij\delta}^\beta \omega_{k\ell\alpha}^\delta \right) \right). \end{aligned}$$

Using the symmetries of the curvature tensor and ω it follows that $g^{il} g^{jk} g^{pq} \text{Rm}_{ijlq} \omega_{kp\alpha}^\beta = 0$. Thus we conclude that

$$g^{il} g^{jk} \nabla_i \nabla_j \omega_{kl\alpha}^\beta = \frac{g^{il} g^{jk}}{2} \left(F_{ij\alpha}^\delta \omega_{k\ell\delta}^\beta + F_{ij\delta}^\beta \omega_{k\ell\alpha}^\delta \right).$$

Thus (2.4) follows, from which the claim $D^* D^* F = 0$ immediately follows. \square

2.2. Hamilton's monotonicity. In [7, 16, 2] entropy functionals were defined which are monotone along Yang-Mills flow. This entropy involves integrating the density $|F_\nabla|^2$ against a solution to the backwards heat equation. In what follows we rederive this monotonicity formula. For concreteness, given a solution to Yang-Mills flow on $[0, T)$, and a final value G_T we consider a one-parameter family

$$(2.5) \quad \begin{cases} \frac{\partial G}{\partial t} &= -\Delta G \\ G(T) &= G_T. \end{cases}$$

As usual we will frequently let G_T be a Dirac delta mass centered at some point of interest.

Lemma 2.2. *Let ∇_t be a solution to Yang-Mills flow and $G_t \in C^\infty(M)$ a solution to (2.5). The following equality holds:*

$$(2.6) \quad \begin{aligned} \frac{\partial}{\partial t} [|F|^2 G] + 4 \left| \frac{\nabla G \lrcorner F}{G} - D^* F \right|^2 G + 4 \nabla^* X_G(\nabla) \\ - 4g^{ip}g^{jq}g^{rs}F_{pr\alpha}^\beta F_{qs\beta}^\alpha \left((\nabla_i \nabla_j G) - \frac{(\nabla_i G)(\nabla_j G)}{G} \right) = 0. \end{aligned}$$

where $X_G(\nabla) := \frac{1}{4}|F|^2(\nabla G) + (\nabla G \lrcorner F)\#F - D^*F\#F$.

Proof. Differentiating $|F|^2 G$ yields that

$$\frac{\partial}{\partial t} [|F|^2 G] = \frac{\partial}{\partial t} [|F|^2] G + |F|^2 \left(\frac{\partial G}{\partial t} \right).$$

For the first term on the right, since ∇ satisfies Yang-Mills flow, $\dot{\Gamma}_{j\beta}^\alpha = g^{uv}\nabla_u F_{vj\beta}^\alpha$, so we compute, while incorporating divergence terms, the quantity

$$\begin{aligned} (\partial_t |F|^2)G &= 2\langle \partial_t F, F \rangle G \\ &= 2\langle D\dot{\Gamma}, F \rangle G \\ &= -2g^{ip}g^{jq}(D_i \dot{\Gamma}_{j\alpha}^\beta)F_{pq\beta}^\alpha G \\ &= -2g^{ip}g^{jq} \left((\nabla_i \dot{\Gamma}_{j\alpha}^\beta)F_{pq\beta}^\alpha - (\nabla_j \dot{\Gamma}_{i\alpha}^\beta)F_{pq\beta}^\alpha \right) G \\ &= -4g^{ip}g^{jq}(\nabla_i \dot{\Gamma}_{j\alpha}^\beta)F_{pq\beta}^\alpha G \\ &= -4g^{ip}g^{jq} \left(\nabla_i \left(\dot{\Gamma}_{j\alpha}^\beta F_{pq\beta}^\alpha G \right) - \dot{\Gamma}_{j\alpha}^\beta (\nabla_i F_{pq\beta}^\alpha)G - \dot{\Gamma}_{j\alpha}^\beta F_{pq\beta}^\alpha (\nabla_i G) \right) \\ &= 4g^{ip}g^{jq}g^{vw} \left((\nabla_v F_{wj\alpha}^\beta)(\nabla_i F_{pq\beta}^\alpha)G + (\nabla_v F_{wj\alpha}^\beta)F_{pq\beta}^\alpha (\nabla_i G) \right) \\ &\quad - 4g^{ip}g^{jq}g^{vw} \left(\nabla_i \left((\nabla_v F_{wj\alpha}^\beta)F_{pq\beta}^\alpha G \right) \right). \end{aligned}$$

For the next term we have, using the second Bianchi identity and multiple insertions of divergence terms,

$$\begin{aligned}
|F|^2 \partial_t G &= -g^{ip} g^{jq} F_{ij\alpha}^\beta F_{pq\beta}^\alpha (-\Delta G) \\
&= g^{ip} g^{jq} F_{ij\alpha}^\beta F_{pq\beta}^\alpha (g^{vw} \nabla_v \nabla_w G) \\
&= g^{ip} g^{jq} g^{vw} \left(\nabla_v \left(F_{ij\alpha}^\beta F_{pq\beta}^\alpha (\nabla_w G) \right) - 2(\nabla_v F_{ij\alpha}^\beta) F_{pq\beta}^\alpha (\nabla_w G) \right) \\
&= g^{ip} g^{jq} g^{vw} \left(\nabla_v \left(F_{ij\alpha}^\beta F_{pq\beta}^\alpha (\nabla_w G) \right) + 2(\nabla_i F_{jv\alpha}^\beta + \nabla_j F_{vi\alpha}^\beta) F_{pq\beta}^\alpha (\nabla_w G) \right) \\
&= g^{ip} g^{jq} g^{vw} \left(\nabla_v \left(F_{ij\alpha}^\beta F_{pq\beta}^\alpha (\nabla_w G) \right) + 4(\nabla_i F_{jv\alpha}^\beta) F_{pq\beta}^\alpha (\nabla_w G) \right) \\
&= g^{ip} g^{jq} g^{vw} \left(\nabla_v \left(F_{ij\alpha}^\beta F_{pq\beta}^\alpha (\nabla_w G) \right) + 4\nabla_i \left(F_{jv\alpha}^\beta F_{pq\beta}^\alpha (\nabla_w G) \right) \right. \\
&\quad \left. - 4g^{ip} g^{jq} g^{vw} \left(F_{jv\alpha}^\beta (\nabla_i F_{pq\beta}^\alpha) (\nabla_w G) + F_{jv\alpha}^\beta F_{pq\beta}^\alpha (\nabla_i \nabla_w G) \right) \right).
\end{aligned}$$

We combine the identities and sort out the divergence terms (Div) with some reindexing,

$$\begin{aligned}
\text{Div} &= g^{ip} g^{jq} g^{vw} \left(\nabla_v \left(F_{ij\alpha}^\beta F_{pq\beta}^\alpha (\nabla_w G) \right) + 4\nabla_i \left(F_{jv\alpha}^\beta F_{pq\beta}^\alpha (\nabla_w G) \right) + 4 \left(\nabla_i \left((\nabla_w F_{jv\alpha}^\beta) F_{pq\beta}^\alpha G \right) \right) \right) \\
&= g^{ip} g^{jq} g^{vw} \left(\nabla_v \left(F_{ij\alpha}^\beta F_{pq\beta}^\alpha (\nabla_w G) \right) + 4\nabla_v \left(F_{ji\alpha}^\beta F_{wq\beta}^\alpha (\nabla_p G) \right) + 4 \left(\nabla_v \left((\nabla_p F_{ji\alpha}^\beta) F_{wq\beta}^\alpha G \right) \right) \right) \\
&= g^{ip} g^{jq} g^{vw} \left(\nabla_v \left(F_{ij\alpha}^\beta F_{pq\beta}^\alpha (\nabla_w G) \right) + 4\nabla_v \left(F_{ij\alpha}^\beta F_{qw\beta}^\alpha (\nabla_p G) \right) + 4 \left(\nabla_v \left((\nabla_p F_{ij\alpha}^\beta) F_{qw\beta}^\alpha G \right) \right) \right).
\end{aligned}$$

Therefore in coordinate invariant form,

$$\text{Div} = -4\nabla^* \left(-\frac{1}{4} |F|^2 \nabla G + (\nabla G \lrcorner F) \# F - D^* F \# F \right).$$

We set $X_G(\nabla) := -\frac{1}{4} |F|^2 (\nabla G) + (\nabla G \lrcorner F) \# F - D^* F \# F$. Combining all terms we have

$$\begin{aligned}
\frac{\partial}{\partial t} [|F|^2 G] &= 4g^{ip} g^{jq} g^{uv} \left((\nabla_u F_{vj\beta}^\alpha) (\nabla_i F_{pq\alpha}^\beta) G + 2(\nabla_u F_{vj\beta}^\alpha) F_{pq\alpha}^\beta (\nabla_i G) + F_{vj\alpha}^\beta F_{pq\beta}^\alpha (\nabla_i \nabla_u G) \right) - 4\nabla^* X_G(\nabla) \\
&= -4 |D^* F|^2 + 8 \langle D^* F, \nabla G \lrcorner F \rangle + 4g^{ip} g^{jq} g^{uv} F_{vj\alpha}^\beta F_{pq\beta}^\alpha (\nabla_i \nabla_u G) - 4(\nabla^* X_G(\nabla)).
\end{aligned}$$

We recombine terms and observe

$$\left| \frac{\nabla G \lrcorner F}{G} - D^* F \right|^2 = \left| \frac{(\nabla G) \lrcorner F}{G} \right|^2 - 2 \left\langle D^* F, \frac{\nabla G \lrcorner F}{G} \right\rangle + |D^* F|^2.$$

Therefore we incorporate this in and have

$$\begin{aligned}
\frac{\partial}{\partial t} (|F|^2 G) &= -4 \left| \frac{\nabla G \lrcorner F}{G} - D^* F \right|^2 + 4 \left| \frac{(\nabla G) \lrcorner F}{G} \right|^2 \\
&\quad + 4g^{ip} g^{jq} g^{uv} F_{vj\alpha}^\beta F_{pq\beta}^\alpha (\nabla_i \nabla_u G) - 4(\nabla^* X_G(\nabla)).
\end{aligned}$$

The result follows. \square

Corollary 2.3. *Let ∇_t be a solution to Yang-Mills flow and $G_t \in C^\infty(M)$ a solution to (2.5). The following equality holds:*

$$\begin{aligned}
\frac{\partial}{\partial t} \left[\int_M |F|^2 G dV_g \right] &+ 4 \int_M \left| \frac{\nabla G \lrcorner F}{G} - D^* F \right|^2 G dV_g \\
&- 4 \int_M g^{ip} g^{jq} g^{rs} F_{pr\alpha}^\beta F_{qs\beta}^\alpha \left((\nabla_i \nabla_j G) - \frac{(\nabla_i G)(\nabla_j G)}{G} \right) dV_g = 0.
\end{aligned}$$

Corollary 2.4 (Hamilton's Entropy Monotonicity Formula, [7] Theorem C). *Let ∇_t be a solution to Yang-Mills flow and $G_t \in C^\infty(M)$ a solution to (2.5). Then*

$$(2.7) \quad 0 = \frac{\partial}{\partial t} \left[(T-t)^2 \int_M |F|^2 G dV_g \right] + 4(T-t)^2 \int_M \left| \frac{\nabla G \lrcorner F}{G} - D^* F \right|^2 G dV_g \\ - 4(T-t)^2 \int_M g^{ip} g^{jq} g^{rs} F_{pr\alpha}^\beta F_{qs\beta}^\alpha \left((\nabla_i \nabla_j G) - \frac{(\nabla_i G)(\nabla_j G)}{G} + \frac{G g_{ij}}{2(T-t)} \right) dV_g.$$

This monotonicity formula is used in proving the following result of Hamilton:

Theorem 2.5 (Hamilton's Monotonicity Formula, [7] Theorem C). *Given the functional*

$$\mathcal{F}(\nabla, t) := (T-t)^2 \int_M |F_\nabla|^2 G dV_g,$$

suppose ∇_t is a solution to Yang-Mills flow on $t \in [0, T)$. Then $\mathcal{F}(\nabla_t, t)$ is monotone decreasing in t when M is Ricci parallel with weakly positive sectional curvatures, while on a general manifold

$$\mathcal{F}(\nabla_t, t) \leq C_M \mathcal{F}(\nabla_\tau, \tau) + C_M (t - \tau)^2 \mathcal{YM}(\nabla_0).$$

whenever $T-1 \leq \tau \leq t \leq T$, and C_M is a constant depending only on M .

2.3. Monotone entropy functionals on \mathbb{R}^n . In this subsection we specialize Hamilton's monotonicity formula [7] to the case of \mathbb{R}^n (compare [2, 12]). We also observe the existence of “steady” and “expanding” entropy functionals which are fixed on steady and expanding solitons respectively. These functionals are so far only formal objects, as they involve integrals which are not likely to converge in general. We will verify the corresponding monotonicity formulas in Proposition 2.7.

Definition 2.6. Let $M = \mathbb{R}^n$, ∇ a connection, and $x_0 \in \mathbb{R}^n$. The *shrinker kernel* based at (x_0, t_0) is given by, for $t < t_0$,

$$(2.8) \quad G_{x_0, t_0}(x, t) := \frac{e^{-\frac{|x-x_0|^2}{4(t_0-t)}}}{(4\pi(t_0-t))^{n/2}},$$

and the \mathcal{F} -functional is given by

$$(2.9) \quad \mathcal{F}_{x_0, t_0}(\nabla, t) := (t - t_0)^2 \int_{\mathbb{R}^n} |F_\nabla|^2 G_{x_0, t_0}(x, t) dV.$$

The *translator kernel* based at (x_0, t_0) is given by, for $t, t_0 \in \mathbb{R}$

$$(2.10) \quad G_{x_0, t_0}^\mathcal{T}(x, t) := e^{\langle x_0, x \rangle - |x_0|^2(t-t_0)},$$

and the $\mathcal{F}^\mathcal{T}$ -functional will be given by

$$(2.11) \quad \mathcal{F}_{x_0, t_0}^\mathcal{T}(\nabla, t) := \int_{\mathbb{R}^n} |F_\nabla|^2 G_{x_0, t_0}^\mathcal{T}(x, t) dV.$$

The *expander kernel* based at (x_0, t_0) is given by, for $t > t_0$,

$$(2.12) \quad G_{x_0, t_0}^\mathcal{E}(x, t) := \frac{e^{\frac{|x-x_0|^2}{4(t-t_0)}}}{(4\pi(t-t_0))^{n/2}},$$

and the $\mathcal{F}^\mathcal{E}$ -functional will be given by

$$(2.13) \quad \mathcal{F}_{x_0, t_0}^\mathcal{E}(\nabla, t) := (t - t_0)^2 \int_{\mathbb{R}^n} |F_\nabla|^2 G_{x_0, t_0}^\mathcal{E}(x, t) dV.$$

Proposition 2.7 (Monotonicity formulas). *Let $\alpha, \beta \in [-\infty, \infty]$ with $\alpha < \beta$, and let $\nabla_t \in \mathcal{A}_E \times [\alpha, \beta]$ be a solution to Yang-Mills flow. Given $(x_0, t_0) \in \mathbb{R}^n \times [\alpha, \beta]$, the functionals $\mathcal{F}_{x_0, t_0}(\nabla_t, t)$, $\mathcal{F}_{x_0, t_0}^\mathcal{T}(\nabla_t, t)$, $\mathcal{F}_{x_0, t_0}^\mathcal{E}(\nabla_t, t)$ are monotonically decreasing in t .*

Proof. We first demonstrate that $G, G^\mathcal{T}$ and $G^\mathcal{E}$ all satisfy (2.5). First we have, for the shrinker kernel (2.8),

$$\begin{aligned} \left(\Delta + \frac{\partial}{\partial t} \right) [G_{x_0, t_0}(x, t)] &= \frac{1}{(4\pi(t_0 - t))^{\frac{n}{2}}} \nabla_i \left(-\frac{(x - x_0)_i}{2(t_0 - t)} e^{-\frac{|x - x_0|^2}{4(t_0 - t)}} \right) \\ &\quad - G_{x_0, t_0}(x, t) \left(\frac{|x - x_0|^2}{4(t_0 - t)^2} - \frac{n}{2(t_0 - t)} \right) \\ &= 0. \end{aligned}$$

This verifies the first case. We next compute, for the translator kernel (2.10),

$$\left(\Delta + \frac{\partial}{\partial t} \right) [G_{x_0, t_0}^\mathcal{T}(x, t)] = \nabla_i (x_i G_{x_0, t_0}^\mathcal{T}(x, t)) - |x_0|^2 G_{x_0, t_0}^\mathcal{T}(x, t) = 0.$$

This verifies the second case. Finally we consider the expander kernel (2.12),

$$\begin{aligned} \left(\Delta + \frac{\partial}{\partial t} \right) [G_{x_0, t_0}^\mathcal{E}(x, t)] &= \frac{1}{(4\pi(t - t_0))^{\frac{n}{2}}} \nabla_i \left(\frac{(x - x_0)_i}{2(t - t_0)} e^{\frac{|x - x_0|^2}{4(t - t_0)}} \right) \\ &\quad - G_{x_0, t_0}^\mathcal{E}(x, t) \left(\frac{n}{2(t - t_0)} + \frac{|x - x_0|^2}{4(t - t_0)^2} \right) \\ &= 0. \end{aligned}$$

We apply Corollary 2.4, and show the vanishing of the last term of (2.7). It suffices to compute, for $\mathcal{X} \in \{\mathcal{E}, \mathcal{T}\}$ and omitted, quantities of the form

$$(2.14) \quad \nabla_i \nabla_j (G_{x_0, t_0}^\mathcal{X}) - \frac{(\nabla_i G_{x_0, t_0}^\mathcal{X})(\nabla_j G_{x_0, t_0}^\mathcal{X})}{G_{x_0, t_0}^\mathcal{X}}.$$

In each case, we compute the two main quantities of the last quantity of (2.7). We first address the shrinker kernel (2.8).

$$\begin{aligned} \nabla_i \nabla_j (G_{x_0, t_0}) &= \frac{1}{(4\pi(t_0 - t))^{\frac{n}{2}}} \nabla_i \nabla_j \left(e^{-\frac{|x - x_0|^2}{4(t_0 - t)}} \right) \\ (2.15) \quad &= \frac{1}{(4\pi(t_0 - t))^{\frac{n}{2}}} \nabla_i \left(-\frac{(x - x_0)_j}{2(t_0 - t)} e^{-\frac{|x - x_0|^2}{4(t_0 - t)}} \right) \\ &= \left(-\frac{\delta_{ij}}{2(t_0 - t)} + \frac{(x - x_0)_i (x - x_0)_j}{4(t_0 - t)^2} \right) G_{x_0, t_0}. \end{aligned}$$

We also have

$$(2.16) \quad \frac{(\nabla_i G_{x_0, t_0})(\nabla_j G_{x_0, t_0})}{G_{x_0, t_0}} = \frac{(x - x_0)_i (x - x_0)_j}{4(t_0 - t)^2} G_{x_0, t_0}.$$

Combining (2.15) and (2.16) yields

$$\nabla_i \nabla_j (G_{x_0, t_0}) - \frac{(\nabla_i G_{x_0, t_0})(\nabla_j G_{x_0, t_0})}{G_{x_0, t_0}} = -\frac{\delta_{ij}}{2(t_0 - t)} G_{x_0, t_0}.$$

We conclude that the last term of (2.7) vanishes, and thus the temporal monotonicity of \mathcal{F}_{x_0, t_0} with respect to t holds.

We next consider the translator kernel (2.10). Observe that

$$\nabla_i \nabla_j (G_{x_0, t_0}^{\mathcal{T}}) - \frac{(\nabla_i G_{x_0, t_0}^{\mathcal{T}})(\nabla_j G_{x_0, t_0}^{\mathcal{T}})}{G_{x_0, t_0}^{\mathcal{T}}} = \nabla_i ((x_0)_j G_{x_0, t_0}^{\mathcal{T}}) - (x_0)_i (x_0)_j G_{x_0, t_0}^{\mathcal{T}} = 0.$$

It follows that the last term in the expression of Corollary 2.3 vanishes, and so the monotonicity of $\mathcal{F}_{x_0, t_0}^{\mathcal{T}}$ holds. Lastly we consider the expander kernel (2.12). Observe that

$$\begin{aligned} \nabla_i \nabla_j (G_{x_0, t_0}^{\mathcal{E}}) &= \frac{1}{(4\pi(t-t_0))^{\frac{n}{2}}} \nabla_i \nabla_j \left(e^{\frac{|x-x_0|^2}{4(t-t_0)}} \right) \\ (2.17) \quad &= \frac{1}{(4\pi(t-t_0))^{\frac{n}{2}}} \nabla_i \left(\frac{(x-x_0)_j}{2(t-t_0)} e^{\frac{|x-x_0|^2}{4(t-t_0)}} \right) \\ &= \left(\frac{\delta_{ij}}{2(t-t_0)} + \frac{(x-x_0)_i (x-x_0)_j}{4(t-t_0)^2} \right) G_{x_0, t_0}^{\mathcal{E}}. \end{aligned}$$

We also have that

$$(2.18) \quad \frac{(\nabla_i G_{x_0, t_0}^{\mathcal{E}})(\nabla_j G_{x_0, t_0}^{\mathcal{E}})}{G_{x_0, t_0}^{\mathcal{E}}} = \frac{(x-x_0)_i (x-x_0)_j}{4(t-t_0)^2} G_{x_0, t_0}^{\mathcal{E}}.$$

Combining (2.17) and (2.18) yields

$$\nabla_i \nabla_j (G_{x_0, t_0}^{\mathcal{E}}(x)) - \frac{(\nabla_i G_{x_0, t_0}^{\mathcal{E}})(\nabla_j G_{x_0, t_0}^{\mathcal{E}})}{G_{x_0, t_0}^{\mathcal{E}}} = \frac{\delta_{ij}}{2(t-t_0)} G_{x_0, t_0}^{\mathcal{E}},$$

We conclude the monotonicity of $\mathcal{F}_{x_0, t_0}^{\mathcal{E}}$ from Corollary 2.4. \square

2.4. Entropy and basic properties of shrinkers. Provided the definition of the \mathcal{F} -functional (2.9) given in §2.3, we set $G_0 := (4\pi t_0)^{-n/2} e^{-\frac{|x-x_0|^2}{4t_0}}$, and have

$$\mathcal{F}_{x_0, t_0}(\nabla) := \mathcal{F}_{x_0, t_0}(\nabla, 0) = t_0^2 \int_{\mathbb{R}^n} |F_{\nabla}|^2 G_0 dV.$$

Definition 2.8. For a connection ∇ the *entropy* is given by

$$\lambda(\nabla) = \sup_{x_0 \in \mathbb{R}^n, t_0 > 0} \mathcal{F}_{x_0, t_0}(\nabla).$$

Definition 2.9. Let ∇_t be a smooth one-parameter family of connections on $\mathbb{R}^n \times (-\infty, 0)$. Then ∇_t is a *self-similar solution* if

$$(2.19) \quad D_{\nabla_t}^* F_{\nabla_t} - \frac{x}{2t} \lrcorner F_{\nabla_t} = 0.$$

Proposition 2.10. Suppose ∇_t is a self-similar solution to Yang-Mills flow on $\mathbb{R}^n \times (-\infty, 0)$. Then there exists an exponential gauge for ∇ such that the connection coefficients satisfy

$$(2.20) \quad \Gamma(x, t) = \frac{1}{\sqrt{-t}} \Gamma\left(\frac{x}{\sqrt{-t}}, -1\right).$$

The exponential gauge is unique up to initial choice of frame at $x = 0$. This proposition is a consequence of two lemmas from [20]. For the first statement we refer the reader directly to the text.

Lemma 2.11 (end of Theorem 3.1 [20], pp.8). *A solution ∇_t to Yang-Mills flow is furthermore a solution to the differential equation*

$$(2.21) \quad \frac{\partial \nabla_t}{\partial t} + \frac{x}{2t} \lrcorner F_{\nabla_t} = 0,$$

if and only if in an exponential gauge the connection matrices satisfy

$$(2.22) \quad \frac{\partial \Gamma_t}{\partial t} + \left(\frac{x}{2t} \lrcorner \partial \Gamma_t \right) + \frac{1}{2t} \Gamma_t = 0.$$

The second lemma demonstrates the equivalences of a characteristic scaling law of connections with (2.22). We will include a detailed proof for convenience.

Lemma 2.12 ([20] Lemma 3.2). *Let the one-parameter family ∇_t with connection coefficient matrices Γ_t be a solution to Yang-Mills flow. Then ∇_t satisfies*

$$(2.23) \quad \frac{\partial \Gamma_t}{\partial t} + \left(\frac{x}{2t} \lrcorner \partial \Gamma_t \right) + \frac{1}{2t} \Gamma_t = 0,$$

on $\mathbb{R}^n \times (-\infty, 0)$, if and only if for all $\lambda \neq 0$,

$$(2.24) \quad \Gamma(x, t) = \lambda \Gamma(\lambda x, \lambda^2 t).$$

Proof. Assuming ∇ satisfies (2.24), we differentiate (2.24) with respect to λ and then evaluate at $\lambda = 1$:

$$\begin{aligned} 0 &= \frac{\partial}{\partial \lambda} (\lambda \Gamma(\lambda x, \lambda^2 t) - \Gamma(x, t)) \Big|_{\lambda=1} \\ &= \Gamma + \lambda^2 x^k \partial_k \Gamma + 2\lambda t \partial_t \Gamma \Big|_{\lambda=1} \\ &= \Gamma + x^k \partial_k \Gamma + 2t \partial_t \Gamma. \end{aligned}$$

Dividing by $2t$ produces the desired result. Next, we show that a solution of (2.23) must consequently satisfy the scaling law (2.24). To do so, we let $\tilde{\nabla}$ be a solution to Yang-Mills flow satisfying (2.24), so that $\tilde{\nabla}$ is a solution to (2.23) with connection coefficient matrix Γ , and let ∇ be yet another solution to (2.23) with connection coefficient matrix $\tilde{\Gamma}$. Set $\Upsilon_t := \Gamma_t - \tilde{\Gamma}_t$. Note that for each t , Υ_t is in the kernel of the following operator

$$\Phi : \Lambda^1(\text{End } E) \rightarrow \Lambda^1(\text{End } E) : B \mapsto \frac{\partial B}{\partial t} + \left(\frac{x}{2t} \lrcorner \partial B \right).$$

We first verify that for any $s \in (-\infty, 0)$ the hypersurface $\mathbb{R}^n \times \{s\}$ is non-characteristic with respect to the operator Φ . This is equivalent to showing that the symbol is non degenerate in the transverse direction of the boundary of $\mathbb{R}^n \times \{s\}$, that is, that

$$\langle \sigma[\Phi], \partial_t \rangle \neq 0.$$

Given that

$$(\sigma[\Phi](B))_{k\alpha}^\beta = \xi_t B + \left(\frac{x}{2t} \lrcorner \xi_x B \right),$$

then we have that

$$\langle \sigma[\Phi], \xi_t \rangle = |\xi_t|^2 \neq 0.$$

Thus, by Holmgren's Uniqueness Theorem (cf. [18] pg. 433), there exists some $\epsilon > 0$ such that on $\mathbb{R}^n \times [s - \epsilon, s + \epsilon]$, we have $\Phi(\Upsilon) = 0$. This demonstrates openness of the set

$$\mathcal{T} := \{\theta \in (-\infty, 0) : \Upsilon_\theta = 0\}.$$

Since this set is closed (the inverse image of zero under a continuous map) by the connectedness of $(-\infty, 0)$, then $\mathcal{T} = (-\infty, 0)$ so we have $\Gamma_t = \tilde{\Gamma}_t$, as desired. The result follows. \square

Proof of Proposition 2.10. Define ∇_t to be a family of connections which are furthermore solutions to Yang-Mills flow with coefficient matrices which satisfy

$$(2.25) \quad \Gamma(x, t) := \left(\frac{1}{\sqrt{-t}} \right) \Gamma \left(\frac{x}{\sqrt{-t}}, -1 \right).$$

We verify that $\Gamma(x, t)$ satisfies the scaling law (2.24) by computation:

$$\begin{aligned}\lambda\Gamma(\lambda x, \lambda^2 t) &= \lambda \frac{1}{\sqrt{-\lambda^2 t}} \left(\frac{\lambda x}{\sqrt{-\lambda^2 t}}, -1 \right) \\ &= \frac{1}{\sqrt{-t}} \Gamma \left(\frac{x}{\sqrt{-t}}, -1 \right) \\ &= \Gamma(x, t).\end{aligned}$$

Thus the scaling law holds. It therefore follows by Lemmas 2.11 and 2.12 that this is equivalent to ∇_t satisfying (2.21). But since ∇_t is a solution to Yang-Mills flow, we conclude that

$$(2.26) \quad D_t^* F_t = \frac{x}{2t} \lrcorner F_t.$$

The result follows. \square

Definition 2.13. A connection ∇ is a *soliton* if, for all $x \in \mathbb{R}^n$,

$$(2.27) \quad D_\nabla^* F_\nabla + \frac{x}{2} \lrcorner F_\nabla = 0.$$

This definition captures the notion of a self-similar solution by considering the $t = -1$ slice. As exhibited in [20], all type I singularities of Yang-Mills flow admit blowup solutions which are nontrivial solitons, thus their study is central to understanding singularity formation of the flow. In this section and the next we collect a number of observations concerning solitons and their structure. First, we observe that solitons can be interpreted as Yang-Mills connections for a certain conformally modified metric on \mathbb{R}^n .

Proposition 2.14. *Suppose ∇ is a soliton on $\mathbb{R}^n, n \geq 5$. Then ∇ is a Yang-Mills connection with respect to the metric $g_{ij} = e^{-\frac{|x|^2}{2(n-4)}} \delta_{ij}$.*

Proof. A calculation shows that for a connection ∇ , Riemannian metric g and function ϕ , one has

$$-D_{e^{2\phi}g}^* F = e^{-2\phi} [-D_g^* F + (n-4)\nabla\phi \lrcorner F].$$

With the choice $\phi = -\frac{|x|^2}{4(n-4)}$, comparing against (2.27) yields the result. \square

Next we establish a number of preliminary properties of solitons and Yang-Mills flow blowups in preparation for understanding the variational properties of the \mathcal{F} -functional. We will use these to show in Corollary 3.7 below that solitons are, after reparameterizing in space and time, the critical points of the \mathcal{F} -functional. First though we consider a more general notion of soliton.

Definition 2.15. The (x_0, t_0) -soliton operator is given by

$$S_{x_0, t_0} : \mathcal{A}_E \rightarrow \Lambda^1(\text{End } E) : \nabla \mapsto D_\nabla^* F_\nabla + \frac{(x - x_0)}{2t_0} \lrcorner F_\nabla.$$

Definition 2.16. Suppose $\nabla \in \ker S_{x_0, t_0}$ so that

$$(2.28) \quad D_\nabla^* F_\nabla + \frac{(x - x_0)}{2t_0} \lrcorner F_\nabla = 0.$$

Then ∇ is called a (x_0, t_0) -soliton.

We next demonstrate the correspondence between the set of $(0, 1)$ -solitons, denoted by \mathfrak{S} , and the set of (x_0, t_0) -solitons, denoted by \mathfrak{S}_{x_0, t_0} .

Lemma 2.17. *For all $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}$, the sets \mathfrak{S}_{x_0, t_0} and \mathfrak{S} are in bijective correspondence.*

Proof. Beginning with $\nabla \in \mathfrak{S}$, we set

$$\tilde{\nabla}(x) := \frac{1}{\sqrt{t_0}} \nabla \left(\frac{x - x_0}{\sqrt{t_0}} \right).$$

Computation yields

$$\begin{aligned} [D_{\tilde{\nabla}} F_{\tilde{\nabla}}]_x &= \frac{1}{t_0^{3/2}} [D_{\nabla} F_{\nabla}]_{\frac{x-x_0}{\sqrt{t_0}}} \\ &= \frac{1}{t_0^{3/2}} \left(\frac{x_0 - x}{2\sqrt{t_0}} \right) \lrcorner [F_{\nabla}]_{\frac{x-x_0}{\sqrt{t_0}}} \\ &= \left(\frac{x_0 - x}{2t_0} \right) \lrcorner \left(\frac{1}{t_0} [F_{\nabla}]_{\frac{x-x_0}{\sqrt{t_0}}} \right) \\ &= \left(\frac{x_0 - x}{2t_0} \right) \lrcorner [F_{\tilde{\nabla}}]_x. \end{aligned}$$

Conversely given $\tilde{\nabla} \in \mathfrak{S}_{x_0, t_0}$, we define $\nabla(x) := \sqrt{t_0} \tilde{\nabla}(x_0 + \sqrt{t_0}x)$. A similar calculation shows that $\nabla \in \mathfrak{S}$. The result follows. \square

2.5. Polynomial energy growth. In the computations to follow deriving the first and second variation of entropy we integrate by parts and encounter many quantities whose integrability is not immediately clear. For this reason we will add an extra condition to the solitons we consider, namely that of “polynomial energy growth,” made precise below. We give a formal argument in Proposition 2.21 showing that blowup limits of Yang-Mills flow, should automatically satisfy this hypothesis. Moreover, for the more delicate analytic arguments we require the curvature itself to be pointwise bounded by some polynomial function. The type I blowup limits constructed in [20] have bounded curvature and so automatically satisfy this hypothesis.

Definition 2.18. A connection ∇ on \mathbb{R}^n has *polynomial energy growth* about $y \in \mathbb{R}^n$ if there exists a polynomial p such that

$$\int_{B_y(r)} |F_{\nabla}|^2 dV \leq p(r).$$

Definition 2.19. A connection ∇ on \mathbb{R}^n has *polynomial curvature growth* if there exists a polynomial p such that for all $x \in \mathbb{R}^n$, one has $|F_{\nabla}|(x) \leq p(r(x))$.

Lemma 2.20. Let $\nabla_t \in \mathcal{A}_E \times [0, T)$ be some solution to Yang-Mills flow on (M^n, g) , with $n \geq 4$. Given $t_1 \in [0, T)$, there exists $R > 0$ and $C = C(t_1, g)$ such that for all $x_0 \in M$, $t \in [t_1, T)$ and $r \leq R$ we have

$$\int_{B_{x_0}(r)} |F_{\nabla_t}|^2 dV_g \leq C e^{1/4} \mathcal{YM}(\nabla_0) r^{n-4} t_1^{\frac{4-n}{2}}.$$

Proof. Let $\rho(x, y)$ denote the distance function on M between x and y , and let G_0^M denote the heat kernel of the manifold M with respect to the metric g based at the center point x_0 at time t_0 . First, using Proposition 2.7 and then appealing to a Euclidean-type heat kernel upper bound, see for instance ([10] Theorem 13.4) we obtain, for any $t < t_0$, for some C_1 dependent

on T and M coming from Theorem 2.5 (which introduces the C_M),

$$\begin{aligned}
 (t_0 - t)^2 \int_{B_{x_0}(r)} |F_t|^2 G_0^M(x, t) dV_g &\leq (t_0 - t)^2 \int_M |F_t|^2 G_0^M(x, t) dV_g \\
 (2.29) \qquad \qquad \qquad &\leq C_M t_0^2 \int_M |F_0|^2 G_0^M(x, 0) dV_g + C_M t_0^2 \mathcal{YM}(\nabla_0) \\
 &\leq C_1 t_0^{\frac{4-n}{2}} \mathcal{YM}(\nabla_0).
 \end{aligned}$$

Also, appealing to a local Euclidean-type heat kernel lower bound, ([10] Theorem 13.8) we have, for sufficiently small R and all $r \leq R$,

$$\begin{aligned}
 (t - t_0)^2 \int_{B_{x_0}(r)} |F_\nabla|^2 G_0^M dV_g &\geq C_2 (t - t_0)^2 \int_{B_{x_0}(r)} |F_t|^2 (4\pi(t_0 - t))^{-\frac{n}{2}} e^{-\frac{\rho(x, x_0)^2}{4(t_0 - t)}} dV_g \\
 (2.30) \qquad \qquad \qquad &= C_2 (t - t_0)^{\frac{4-n}{2}} e^{-\frac{r^2}{4(t_0 - t)}} \int_{B_{x_0}(r)} |F_t|^2 dV_g.
 \end{aligned}$$

By combining inequalities (2.29) and (2.30) and setting $C := \frac{C_1}{C_2}$ we have

$$\begin{aligned}
 (2.31) \qquad \int_{B_{x_0}(r)} |F_t|^2 dV_g &\leq C (t - t_0)^{\frac{n-4}{2}} (t_0)^{\frac{4-n}{2}} e^{\frac{r^2}{4(t_0 - t)}} \mathcal{YM}(\nabla_0) \\
 &= C e^{\frac{r^2}{4(t_0 - t)}} \left(1 - \frac{t}{t_0}\right)^{\frac{n-4}{2}} \mathcal{YM}(\nabla_0).
 \end{aligned}$$

Now take $t_0 = t + r^2$ and observe that

$$\begin{aligned}
 \left(1 - \frac{t}{t + r^2}\right)^{\frac{n-4}{2}} &= \left(\frac{r^2}{t + r^2}\right)^{\frac{n-4}{2}} \\
 (2.32) \qquad \qquad \qquad &= (r^2)^{\frac{n-4}{2}} (t + r^2)^{-\frac{n-4}{2}} \\
 &\leq r^{n-4} t^{-\frac{n-4}{2}} \\
 &\leq r^{n-4} (t_1)^{\frac{4-n}{2}}.
 \end{aligned}$$

Applying (2.32) in (2.31) we conclude

$$(2.33) \qquad \int_{B_{x_0}(r)} |F_t|^2 dV_g \leq C e^{1/4} \mathcal{YM}(\nabla_0) r^{n-4} t_1^{\frac{4-n}{2}}.$$

The result follows. \square

Proposition 2.21. *Let ∇_t be a solution to Yang-Mills flow on (M, g) which exists for $t \in [0, T)$. Fix some local framing and let Γ denote the coefficient matrix of ∇ , and let $\nabla_s^{r_i}(y)$ be the connection with coefficient matrix $\Gamma_s^{r_i}(y) := r_i \Gamma(\exp_{x_0}(r_i y), T + r_i^2 s)$. Assume that $\nabla_s^{r_i}(y)$ converges strongly on $M \times (-\infty, 0)$ as $r_i \rightarrow 0$ to a self-similar solution ∇_s^∞ . Then for any $r > 0$ we have*

$$(2.34) \qquad \int_{B_0(r)} |F_{\nabla^\infty}(y, -1)|^2 dy \leq e^{1/4} \mathcal{YM}\left(\frac{T}{2}\right)^{-\frac{n-4}{2}} r^{n-4}.$$

Proof. For each $i \in \mathbb{N}$ the following equality holds

$$(2.35) \qquad \int_{B_0(r)} |F_{\nabla^{r_i}}(y, s)|^2 dV_y = \int_{B_{x_0}(rr_i)} r_i^4 |F_\nabla(\exp_{x_0}(r_i x), T + r_i^2 s)|^2 r_i^{-n} dV_x.$$

We look at the temporal slice $s = -1$ and choosing i sufficiently large to ensure that $r_i^2 \leq \frac{T}{2}$. Then applying Lemma 2.20 we have

$$(2.36) \quad \int_{B_0(r)} |F_{\nabla^{r_i}}(y, -1)|^2 dV_y \leq 2e^{1/4} \mathcal{YM}(\nabla_0) \left(\frac{T}{2}\right)^{-\frac{n-4}{2}} r^{n-4},$$

sending $i \rightarrow \infty$ yields the result. \square

3. VARIATIONAL PROPERTIES

In this section we establish some fundamental variational properties of the Yang-Mills entropy. We begin by establishing first and second variation formulas for the entropy functional \mathcal{F} , including variations of the point in spacetime. We begin with some preliminary integration by parts formulas, then use these to obtain the first and second variations. These yield as corollaries that solitons are characterized as critical points for the \mathcal{F} -functional. We combine these calculations in §5 to establish that the entropy is indeed achieved for a soliton with polynomial energy growth, realized by the \mathcal{F} -functional based at the basepoint of the given soliton. Moreover this point uniquely realizes the entropy, unless the soliton exhibits some flat directions.

3.1. Preliminary calculations.

Lemma 3.1. *Let ∇^∞ satisfy (2.27) with polynomial energy growth. Then setting $\nabla(x) := \frac{1}{\sqrt{t_0}} \nabla^\infty \left(\frac{x-x_0}{\sqrt{t_0}} \right)$, it holds that for all $\theta \geq 0$,*

$$(3.1) \quad I_\theta(\nabla) := \int_{\mathbb{R}^n} |x - x_0|^\theta |F_\nabla|^2 e^{\frac{-|x-x_0|^2}{4t_0}} dV < \infty.$$

Proof. We observe that ∇ as defined above blows up at (x_0, t_0) , and, via change of variables, satisfies

$$\begin{aligned} \int_{B_{x_0}(r)} |F_\nabla|^2 dV &= \int_{B_{x_0}\left(\frac{r}{\sqrt{t_0}}\right)} |F_{\nabla^\infty}|^2 t_0^{\frac{n-2}{2}} dV \\ &\leq p\left(\frac{r}{\sqrt{t_0}}\right) t_0^{\frac{n-2}{2}}. \end{aligned}$$

For each $r \in \mathbb{R}$, set $A_{x_0}(r) := B_{x_0}(r)/B_{x_0}(r-1)$. Then partitioning \mathbb{R}^n into a union of annuli yields

$$\begin{aligned} I_\theta &= \sum_{r=1}^{\infty} \int_{A_{x_0}(r)} |x - x_0|^\theta |F|^2 e^{\frac{-|x-x_0|^2}{4t_0}} dV \\ &\leq \sum_{r=1}^{\infty} r^\theta e^{\frac{(r-1)^2}{4t_0}} \int_{A_{x_0}(r)} |F|^2 dV \\ &\leq \sum_{r=1}^{\infty} \int_{B_{x_0}(r)} |F|^2 dV. \end{aligned}$$

Incorporating the assumption of polynomial energy growth (3.1), we conclude that

$$(3.2) \quad I_\theta \leq \sum_{r=1}^{\infty} r^\theta e^{\frac{-(k-1)^2}{4t_0}} p\left(\frac{r}{\sqrt{t_0}}\right) t_0^{\frac{n-2}{2}} < \infty.$$

The result follows. \square

Lemma 3.2. *Let ∇ be a (χ, τ) -soliton with polynomial energy growth and let $G_0 = e^{-\frac{|x-x_0|^2}{4t_0}}$. Then for any vector fields $\xi = \xi^i \partial_i$ such that $|\xi|^2 G_0 \in L^\infty(\mathbb{R}^n)$,*

$$(3.3) \quad \int_{\mathbb{R}^n} \xi^i (x - x_0)^i |F|^2 G_0 dV = 8t_0 \int_{\mathbb{R}^n} F_{pu\alpha}^\beta F_{iu\beta}^\alpha \left(\partial_p \xi^i + \frac{1}{2} \left(\frac{x_0}{t_0} - \frac{\chi}{\tau} + x \left(\frac{1}{\tau} - \frac{1}{t_0} \right) \right)^p \xi^i \right) G_0 dV \\ + 2t_0 \int_{\mathbb{R}^n} (\partial_i \xi^i) |F|^2 G_0 dV.$$

Proof. We let $\xi = \xi^i \partial_i$ be a smooth vector field on $M = \mathbb{R}^n$ and $\eta \in C_c^\infty(\mathbb{R}^n)$ with $|\eta| \leq 1$. Observe that

$$(3.4) \quad \frac{\partial G_0}{\partial x^i} = \frac{-(x - x_0)^i}{2t_0} G_0.$$

Applying this equality and integrating by parts we obtain

$$(3.5) \quad \int_{\mathbb{R}^n} \xi^i (x - x_0)^i |F|^2 \eta G_0 dV = -2t_0 \int_{\mathbb{R}^n} \xi^i |F|^2 (\partial_i G_0) \eta dV \\ = 2t_0 \int_{\mathbb{R}^n} (\partial_i (\xi^i |F|^2 \eta)) G_0 dV \\ = 2t_0 \int_{\mathbb{R}^n} ((\partial_i \xi^i \eta) |F|^2 + \eta \xi^i (\partial_i |F|^2)) G_0 dV.$$

Additionally by an application of the Bianchi identity we have

$$\int_{\mathbb{R}^n} \langle D^* F, \xi \lrcorner F \rangle \eta G_0 dV = \int_{\mathbb{R}^n} \left((\nabla_p F_{pu\alpha}^\beta) F_{iu\beta}^\alpha \right) \xi^i \eta G_0 dV \\ = \int_{\mathbb{R}^n} \left(\nabla_p \left(F_{pu\alpha}^\beta F_{iu\beta}^\alpha \right) - F_{pu\alpha}^\beta (\nabla_p F_{iu\beta}^\alpha) \right) \xi^i \eta G_0 dV \\ = - \int_{\mathbb{R}^n} \left(F_{pu\alpha}^\beta F_{iu\beta}^\alpha \right) \nabla_p (\xi^i \eta G_0) dV + \int_{\mathbb{R}^n} F_{pu\alpha}^\beta (\nabla_u F_{pi\beta}^\alpha + \nabla_i F_{up\beta}^\alpha) \xi^i \eta G_0 dV \\ = - \int_{\mathbb{R}^n} \left(\left(F_{pu\alpha}^\beta F_{iu\beta}^\alpha \right) \left((\partial_p (\eta \xi^i)) - \frac{(x - x_0)^p}{2t_0} \eta \xi^i \right) \right) G_0 dV \\ + \int_{\mathbb{R}^n} \left(F_{pu\alpha}^\beta (\nabla_u F_{pi\beta}^\alpha) \right) \xi^i \eta G_0 dV - \frac{1}{2} \int_{\mathbb{R}^n} \left(\nabla_i (F_{pu\alpha}^\beta F_{pu\beta}^\alpha) \right) \xi^i \eta G_0 dV \\ = - \int_{\mathbb{R}^n} \left(\left(F_{pu\alpha}^\beta F_{iu\beta}^\alpha \right) \left((\partial_p (\eta \xi^i)) - \frac{(x - x_0)^p}{2t_0} \eta \xi^i \right) \right) G_0 dV \\ + \int_{\mathbb{R}^n} \left(F_{pu\alpha}^\beta (\nabla_u F_{pi\beta}^\alpha) \right) \xi^i \eta G_0 dV + \frac{1}{2} \int_{\mathbb{R}^n} (\partial_i (|F|^2)) \xi^i \eta G_0 dV.$$

We multiply through the equality by $4t_0$ and isolate the last term on the right. Applying this to (3.5),

$$\int_{\mathbb{R}^n} \xi^i (x - x_0)^i |F|^2 \eta G_0 dV = 4t_0 \int_{\mathbb{R}^n} F_{pu\alpha}^\beta (F_{iu\beta}^\alpha) \left((\partial_p (\eta \xi^i)) - \frac{(x - x_0)^p}{2t_0} \eta \xi^i \right) G_0 dV \\ - 4t_0 \int_{\mathbb{R}^n} \left(F_{pu\alpha}^\beta (\nabla_u F_{pi\beta}^\alpha) + (D^* F)_{u\alpha}^\beta F_{iu\beta}^\alpha \right) \xi^i \eta G_0 dV \\ + 2t_0 \int_{\mathbb{R}^n} (\nabla_i (\xi^i \eta)) |F|^2 G_0 dV.$$

We let η_R be a cut off function with support within $B(R)$ which cuts off to zero linearly between $B(R)$ and $B(R+1)$. Applying this to the above expression and sending $R \rightarrow \infty$, it follows from the Dominated Convergence Theorem (one will see that for each of the test functions η we insert this holds) that we obtain

(3.6)

$$\begin{aligned} \int_{\mathbb{R}^n} \xi^i (x - x_0)^i |F|^2 G_0 dV &= 2t_0 \int_{\mathbb{R}^n} (\partial_i(\xi^i)) |F|^2 G_0 dV + 4t_0 \int_{\mathbb{R}^n} F_{pu\alpha}^\beta (F_{iu\beta}^\alpha) \left((\partial_p(\xi^i)) - \frac{(x - x_0)^p}{2t_0} \xi^i \right) G_0 dV \\ &\quad - 4t_0 \int_{\mathbb{R}^n} \left(F_{pu\alpha}^\beta (\nabla_u F_{pi\beta}^\alpha) + (D^* F)_{u\alpha}^\beta F_{iu\beta}^\alpha \right) \xi^i G_0 dV. \end{aligned}$$

Then we manipulate the latter term of above. Since the integrand consists of an inner product against a skew quantity we may consider the skew projection of ∇F onto proper components. A subsequent application of the Bianchi identity and inclusion of divergence term yields

$$\begin{aligned} \int_{\mathbb{R}^n} \left(F_{pu\alpha}^\beta (\nabla_u F_{pi\beta}^\alpha) \right) \xi^i G_0 dV &= \frac{1}{2} \int_{\mathbb{R}^n} \left(F_{pu\alpha}^\beta (\nabla_u F_{pi\beta}^\alpha - \nabla_p F_{ui\beta}^\alpha) \right) \xi^i G_0 dV \\ &= -\frac{1}{2} \int_{\mathbb{R}^n} \left(F_{pu\alpha}^\beta (\nabla_i F_{up\beta}^\alpha) \right) \xi^i G_0 dV \\ &= -\frac{1}{4} \int_{\mathbb{R}^n} \nabla_i (|F|^2) \xi^i G_0 dV \\ &= \frac{1}{4} \int_{\mathbb{R}^n} |F|^2 \nabla_i (\xi^i G_0) dV \\ &= \frac{1}{4} \int_{\mathbb{R}^n} |F|^2 (\partial_i \xi^i) G_0 dV - \frac{1}{8t_0} \int_{\mathbb{R}^n} |F|^2 \xi^i (x - x_0)^i G_0 dV. \end{aligned}$$

We insert this identity into (3.6) and obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \xi^i (x - x_0)^i |F|^2 G_0 dV &= t_0 \int_{\mathbb{R}^n} (\partial_i(\xi^i)) |F|^2 G_0 dV + 4t_0 \int_{\mathbb{R}^n} F_{pu\alpha}^\beta (F_{iu\beta}^\alpha) \left((\partial_p(\xi^i)) - \frac{(x - x_0)^p}{2t_0} \xi^i \right) G_0 dV \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^n} |F|^2 (x - x_0)^i \xi^i G_0 dV - 4t_0 \int_{\mathbb{R}^n} (D^* F)_{u\alpha}^\beta F_{iu\beta}^\alpha \xi^i G_0 dV. \end{aligned}$$

We note that since ∇ is a self-similar solution based at (χ, τ) we replace $D^* F = \frac{\chi - x}{2\tau} \lrcorner F_\nabla$ to yield

$$\begin{aligned} \int_{\mathbb{R}^n} |F|^2 (x - x_0)^i \xi^i G_0 dV &= 8t_0 \int_{\mathbb{R}^n} F_{pu\alpha}^\beta F_{iu\beta}^{ga} \left(\partial_p \xi^i + \frac{1}{2} \left(\frac{x_0}{t_0} - \frac{\chi}{\tau} - x \left(\frac{1}{t_0} - \frac{1}{\tau} \right) \right)^p \xi^i \right) G_0 dV \\ &\quad + 2t_0 \int_{\mathbb{R}^n} (\partial_i \xi^i) |F|^2 G_0 dV, \end{aligned}$$

as claimed. \square

Corollary 3.3 (Soliton Identities). *Let $\nabla \in \mathfrak{S}$ satisfy polynomial energy growth and set $G_0 := e^{-\frac{|x-x_0|^2}{4t_0}}$. Let $V = V^i \partial_i$ be any constant vector field on \mathbb{R}^n . Then the following equalities hold.*

- (a) $\int_{\mathbb{R}^n} \left(\frac{|x-x_0|^2}{4} + \frac{(4-n)}{2} \right) |F|^2 G_0 dV = - \int_{\mathbb{R}^n} \langle (x(t_0 - 1) + x_0) \lrcorner F, (x - x_0) \lrcorner F \rangle G_0 dV,$
- (b) $\int_{\mathbb{R}^n} \frac{\langle x - x_0, V \rangle}{2} |F|^2 G_0 dV = -2 \int_{\mathbb{R}^n} \langle (x(t_0 - 1) + x_0) \lrcorner F, V \lrcorner F \rangle G_0 dV.$

Proof. We set $\chi = 0$ and $\tau = 1$ to incorporate the soliton equation (2.27). To obtain (a) we insert $\xi^i = \frac{(x-x_0)^i}{4}$ into (3.3), and for (b) we use $\xi^i = \frac{V^i}{2}$. \square

Corollary 3.4. *Let $\nabla \in \mathfrak{S}_{x_0, t_0}$ be a (x_0, t_0) -soliton with polynomial energy growth. Let $V = V^i \partial_i$ be any constant vector field on \mathbb{R}^n and $\gamma \in [1, n] \cap \mathbb{N}$. Then the following equalities hold.*

- (a) $\int_{\mathbb{R}^n} \left((4-n) + \frac{|x-x_0|^2}{2t_0} \right) |F|^2 G_0 dV = 0,$
- (b) $\int_{\mathbb{R}^n} (x-x_0)^\gamma |F|^2 G_0 dV = 0,$
- (c) $\int_{\mathbb{R}^n} |x-x_0|^4 |F|^2 G_0 dV = 4(n-2)(n-4)t_0^2 \int_{\mathbb{R}^n} |F|^2 G_0 dV - 64t_0^3 \int_{\mathbb{R}^n} |D^* F|^2 G_0 dV,$
- (d) $\int_{\mathbb{R}^n} |x-x_0|^2 \langle V, x-x_0 \rangle |F|^2 G_0 dV = \int_{\mathbb{R}^n} \langle (V \lrcorner F), D^* F \rangle G_0 dV = 0,$
- (e) $\int_{\mathbb{R}^n} \langle V, x-x_0 \rangle^2 |F|^2 G_0 dV = 2t_0 \int_{\mathbb{R}^n} |V|^2 |F|^2 G_0 dV - 8t_0 \int_{\mathbb{R}^n} |V \lrcorner F|^2 G_0 dV.$

Proof. Starting with (3.3), we set $\tau = t_0$ and $\chi = x_0$. This yields

$$(3.7) \quad \int_{\mathbb{R}^n} |F|^2 (x-x_0)^i \xi^i G_0 dV = 8t_0 \int_{\mathbb{R}^n} F_{pu\alpha}^\beta F_{iu\beta}^\alpha (\partial_p \xi^i) G_0 dV + 2t_0 \int_{\mathbb{R}^n} (\partial_i \xi^i) |F|^2 G_0 dV.$$

We approach the listed quantities of the lemma with this identity.

- (a) This immediately follows by setting $\xi^i := \frac{(x-x_0)^i}{4t_0}$.
- (b) This immediately follows by setting $\xi^i := \delta^{\gamma i}$.
- (c) Set $\xi^i := |x-x_0|^2 (x-x_0)^i$. Prior to solving we compute the following derivative:

$$\partial_p (|x-x_0|^2 (x-x_0)^i) = 2(x-x_0)^p (x-x_0)^i + |x-x_0|^2 \delta_{ip}.$$

Applying this to (3.7) gives

$$\begin{aligned} \int_{\mathbb{R}^n} |x-x_0|^4 |F|^2 G_0 dV &= 2t_0 \int_{\mathbb{R}^n} (2+n) |x-x_0|^2 |F|^2 G_0 dV \\ &\quad + 8t_0 \int_{\mathbb{R}^n} F_{pu\alpha}^\beta F_{iu\beta}^\alpha (2(x-x_0)^p (x-x_0)^i + |x-x_0|^2 \delta^{ip}) G_0 dV \\ &= 2(n-2)t_0 \int_{\mathbb{R}^n} |x-x_0|^2 |F|^2 G_0 dV - 16t_0 \int_{\mathbb{R}^n} |F \lrcorner (x-x_0)|^2 G_0 dV. \end{aligned}$$

Now we replace the first term with the identity of (a) and the second term with the (x_0, t_0) -soliton equation (2.28) to conclude that

$$\int_{\mathbb{R}^n} |x-x_0|^4 |F|^2 G_0 dV = 4(n-2)(n-4)t_0^2 \int_{\mathbb{R}^n} |F|^2 G_0 dV - 64t_0^3 \int_{\mathbb{R}^n} |D^* F|^2 G_0 dV.$$

- (d) To prove this identity it will require applying two different test functions to (3.7). First we set $\xi^i := |x-x_0|^2 V^i$. Then using (b) and the (x_0, t_0) -soliton equation (2.28) we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |x-x_0|^2 \langle V, (x-x_0) \rangle |F|^2 G_0 dV &= 4t_0 \int_{\mathbb{R}^n} \langle (x-x_0), V \rangle |F|^2 G_0 dV \\ &\quad + 16t_0 \int_{\mathbb{R}^n} F_{pu\alpha}^\beta F_{iu\beta}^\alpha (x-x_0)^p V^i G_0 dV \\ &= -32t_0^2 \int_{\mathbb{R}^n} (D^* F)_{u\alpha}^\beta F_{iu\beta}^\alpha V^i G_0 dV. \end{aligned}$$

Now we will instead consider $\xi^i := \langle V, (x-x_0) \rangle (x-x_0)^i$. Prior to this we differentiate

$$\partial_p [\langle V, (x-x_0) \rangle (x-x_0)^i] = V^p (x-x_0)^i + \langle V, (x-x_0) \rangle \delta_{ip}.$$

therefore we have, applying (b) to (3.7),

$$\begin{aligned}
\int_{\mathbb{R}^n} \langle V, (x - x_0) \rangle |x - x_0|^2 |F|^2 G_0 dV &= 2t_0 \int_{\mathbb{R}^n} (V^i (x - x_0)^i + n \langle V, (x - x_0) \rangle) |F|^2 G_0 dV \\
&\quad + 8t_0 \int_{\mathbb{R}^n} F_{pu\alpha}^\beta F_{iu\beta}^\alpha (V^p (x - x_0)^i + \langle V, (x - x_0) \rangle \delta^{ip}) G_0 dV \\
&= 8t_0 \int_{\mathbb{R}^n} F_{pu\alpha}^\beta F_{iu\beta}^\alpha (V^p (x - x_0)^i) G_0 dV \\
&= -16t_0^2 \int_{\mathbb{R}^n} (D^* F)_{u\alpha}^\beta F_{pu\beta}^\alpha V^p G_0 dV.
\end{aligned}$$

By equality of the two expressions we conclude that

$$\int_{\mathbb{R}^n} \langle V, (x - x_0) \rangle |x - x_0|^2 |F|^2 G_0 dV = \int_{\mathbb{R}^n} \langle D^* F, F \lrcorner V \rangle G_0 dV = 0.$$

(e) Set $\xi^i := \langle V, x - x_0 \rangle V^i$. This quantity differentiated is precisely

$$\int_{\mathbb{R}^n} \langle V, x - x_0 \rangle^2 |F|^2 G_0 dV = 2t_0 \int_{\mathbb{R}^n} |V|^2 |F|^2 G_0 dV - 8t_0 \int_{\mathbb{R}^n} |F \lrcorner V|^2 G_0 dV.$$

The final result follows, and the proof is complete. \square

One consequence of these identities is that the Yang-Mills flow in dimension $n = 4$ cannot exhibit type I singularities.

Proposition 3.5. *Let $E \rightarrow (M^4, g)$ be a smooth vector bundle, and suppose ∇_t is a solution to Yang-Mills flow on E which exists on a maximal time interval $[0, T)$, with $T < \infty$. Then*

$$\lim_{t \rightarrow T} (T - t) |F_{\nabla_t}| = \infty.$$

Moreover, any soliton on \mathbb{R}^n , $n \leq 4$, is flat.

Proof. Suppose to the contrary that there exists a $C \in \mathbb{R}$ so that

$$\lim_{t \rightarrow T} (T - t) |F_{\nabla_t}| \leq C.$$

By Weinkove's main theorem (pp.2 [20]), we can construct a type I blowup limit ∇_∞ which is a self-similar solution, whose time $t = -1$ slice is a nonflat $(0, 1)$ -soliton. By Corollary 3.4 part (a) since $\dim M = 4$, we conclude

$$(3.8) \quad \int_{\mathbb{R}^n} \frac{|x|^2}{2} |F_{\nabla_\infty}|^2 G_0 dV = 0.$$

Thus it follows that ∇_∞ is flat, but this is a contradiction since by construction ∇_∞ is nonflat. The result follows. \square

3.2. First variation. In this subsection we compute the first variation of the \mathcal{F} -functional. For both the first and second variation computations we the dependence on s will be dropped for all terms except the varying base point (x_s, t_s) during coordinate computations. Moreover, the variational calculations require the variation of the connection to be in a certain weighted Sobolev space (see 4.1) which we suppress here.

Proposition 3.6 (First variation). *Let Γ_s , x_s and t_s be one parameter families of connections, points in \mathbb{R}^n , and positive real numbers respectively, and set*

$$(3.9) \quad G_s(x) := (4\pi t_s)^{\frac{-n}{2}} e^{\frac{-|x - x_s|^2}{4t_s}},$$

and furthermore set

$$\dot{t}_s := \frac{dt_s}{ds}, \quad \dot{x}_s := \frac{dx_s}{ds}, \quad \dot{\Gamma}_s := \frac{d\Gamma_s}{ds}.$$

Then

$$\begin{aligned} \frac{d}{ds} [\mathcal{F}_{x_s, t_s}(\nabla_s)] &= \dot{t}_s \int_{\mathbb{R}^n} \left(t_s \left(\frac{4-n}{2} \right) + \frac{|x-x_s|^2}{4} \right) |F_s|^2 G_s dV \\ &+ t_s \int_{\mathbb{R}^n} \frac{\langle \dot{x}_s, x-x_s \rangle}{2} |F_s|^2 G_s dV \\ &+ 4t_s^2 \int_{\mathbb{R}^n} \left\langle \dot{\Gamma}_s, D_s^* F_s + \left(\frac{(x-x_s)}{2t_s} \lrcorner F_s \right) \right\rangle G_s dV. \end{aligned} \quad (3.10)$$

Proof. We first differentiate the following expression

$$\frac{d}{ds} \left[\int_{\mathbb{R}^n} |F_s|^2 G_s dV \right] = \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial s} |F_s|^2 \right) G_s dV + \int_{\mathbb{R}^n} |F_s|^2 \left(\frac{\partial}{\partial s} G_s \right) dV.$$

We compute the first quantity on the right side of the equality:

$$\begin{aligned} \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial s} |F_s|^2 \right) G_s dV &= -2 \int_{\mathbb{R}^n} \left(\nabla_i \dot{\Gamma}_{j\alpha}^\beta - \nabla_j \dot{\Gamma}_{i\alpha}^\beta \right) F_{ij\beta}^\alpha G dV \\ &= 2 \int_{\mathbb{R}^n} \dot{\Gamma}_{j\alpha}^\beta (\nabla_i F_{ij\beta}^\alpha) G dV + 2 \int_{\mathbb{R}^n} \dot{\Gamma}_{j\alpha}^\beta F_{ij\beta}^\alpha (\nabla_i G) dV \\ &\quad - 2 \int_{\mathbb{R}^n} \dot{\Gamma}_{i\alpha}^\beta (\nabla_j F_{ij\beta}^\alpha) G dV - 2 \int_{\mathbb{R}^n} \dot{\Gamma}_{i\alpha}^\beta F_{ij\beta}^\alpha (\nabla_j G) dV \\ &= 4 \int_{\mathbb{R}^n} \langle \dot{\Gamma}_s, D_s^* F_s \rangle G_s dV + 4 \int_{\mathbb{R}^n} \dot{\Gamma}_{j\alpha}^\beta F_{ij\beta}^\alpha (\nabla_i G) dV. \end{aligned} \quad (3.11)$$

We differentiate G_s and obtain

$$\frac{\partial}{\partial s} [G_s] = \left(\frac{-n}{2} \frac{\dot{t}_s}{t_s} + \frac{\dot{t}_s |x-x_s|^2}{4t_s^2} + \frac{\langle \dot{x}_s, x-x_s \rangle}{2t_s} \right) G_s. \quad (3.12)$$

Furthermore,

$$\nabla_i G = -\frac{(x-x_s)^i}{2t_s} G. \quad (3.13)$$

Applying this to (3.11) gives that

$$\begin{aligned} \int_{\mathbb{R}^n} \dot{\Gamma}_{j\alpha}^\beta F_{ij\beta}^\alpha (\nabla_i G) dV &= \int_{\mathbb{R}^n} \dot{\Gamma}_{j\alpha}^\beta F_{ij\beta}^\alpha \left(-\frac{(x-x_s)^i}{2t_s} \right) G dV \\ &= \int_{\mathbb{R}^n} \left\langle \dot{\Gamma}_s, \left(\frac{(x-x_s)}{2t_s} \lrcorner F_s \right) \right\rangle G_s dV. \end{aligned}$$

So we conclude that

$$\begin{aligned} \frac{d}{ds} \left[\int_{\mathbb{R}^n} |F_s|^2 G_s dV \right] &= \int_{\mathbb{R}^n} |F_s|^2 \left(\frac{-n}{2} \frac{\dot{t}_s}{t_s} + \frac{\dot{t}_s |x-x_s|^2}{4t_s^2} + \frac{\langle \dot{x}_s, x-x_s \rangle}{2t_s} \right) G_s dV \\ &\quad + 4 \int_{\mathbb{R}^n} \left\langle \dot{\Gamma}_s, D_s^* F_s + \left(\frac{(x-x_s)}{2t_s} \lrcorner F_s \right) \right\rangle G_s dV. \end{aligned} \quad (3.14)$$

With this in mind we differentiate the expression

$$\frac{d}{ds} \left[t_s^2 \int_{\mathbb{R}^n} |F_s|^2 G_s dV \right] = 2t_s \dot{t}_s \int_{\mathbb{R}^n} |F_s|^2 G_s dV + t_s^2 \left(\frac{\partial}{\partial s} \left[\int_{\mathbb{R}^n} |F_s|^2 G_s dV \right] \right),$$

and then applying (3.14) we obtain that

$$\begin{aligned} \frac{d}{ds} \left[t_s^2 \int_{\mathbb{R}^n} |F_s|^2 G_s dV \right] &= t_s^2 \int_{\mathbb{R}^n} |F_s|^2 \left(\frac{-n \dot{t}_s}{2 t_s} + \frac{\dot{t}_s |x - x_s|^2}{4 t_s^2} + \frac{\langle \dot{x}_s, x - x_s \rangle}{2 t_s} + \frac{2 \dot{t}_s}{t_s} \right) G_s dV \\ &\quad + 4 t_s^2 \int_{\mathbb{R}^n} \left\langle \dot{\Gamma}_s, D_s^* F_s + \left(\frac{(x - x_s)}{2 t_s} \lrcorner F_s \right) \right\rangle G_s dV. \end{aligned}$$

Reordering terms yields the result. \square

Corollary 3.7. *The point (∇, x_0, t_0) is a critical point of the \mathcal{F} -functional if and only if ∇ is an (x_0, t_0) -soliton.*

Proof. If (∇, x_0, t_0) is a critical point, then all partial derivatives with respect to t , x and Γ vanish. We note that if we vary only the connection coefficient matrix Γ then we have

$$0 = 4 t_s^2 \int_{\mathbb{R}^n} \left\langle \dot{\Gamma}_s, D_s^* F_s + \left(\frac{(x - x_s)}{2 t_s} \lrcorner F_s \right) \right\rangle G_s dV.$$

In particular, this implies that ∇ is an (x_0, t_0) -soliton. Conversely, given a point (x_0, t_0) and that ∇ is a soliton, then we apply identities (a) and (b) of Corollary 3.4 to the variation identity of Proposition 3.6. Each quantity vanishes, yielding that (∇, x_0, t_0) is a critical point and the result follows. \square

Proposition 3.8. *Let $\nabla_t \in \mathcal{A}_E \times [0, T]$ be a solution to Yang-Mills flow with polynomial energy growth. Then $\lambda(\nabla_t)$ is non-increasing in t .*

Proof. Let $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2 < T$. Given $\epsilon > 0$ there exists (x_0, t_0) such that

$$(3.15) \quad \mathcal{F}_{x_0, t_0}(\nabla_{t_2}) \geq \lambda(\nabla_{t_2}) - \epsilon.$$

Thus it follows from Hamilton's monotonicity formula that for any $\delta \in (0, t_2)$ we have

$$(3.16) \quad \mathcal{F}_{x_0, t_0+t_2}(\nabla_{t_2}, t_2) \leq \mathcal{F}_{x_0, t_0+t_2}(\nabla_{t_2-\delta}, t_2 - \delta).$$

But we observe that

$$\begin{aligned} \mathcal{F}_{x_0, t_0+t_2}(\nabla_{t_2}, t_2) &= \int_{\mathbb{R}^n} |F_{\nabla_{t_2}}|^2 e^{-\frac{|x-x_0|^2}{4((t_0+t_2)-t_2)}} dV \\ (3.17) \quad &= \int_{\mathbb{R}^n} |F_{\nabla_{t_2}}|^2 e^{-\frac{|x-x_0|^2}{4t_0}} dV \\ &= \mathcal{F}_{x_0, t_0}(\nabla_{t_2}). \end{aligned}$$

Thus combining (3.16) and (3.17) we have

$$\begin{aligned} \mathcal{F}_{x_0, t_0}(\nabla_{t_2}) &\leq \mathcal{F}_{x_0, t_0+t_2}(\nabla_{t_2-\delta}, t_2 - \delta) \\ (3.18) \quad &= \int_{\mathbb{R}^n} |F_{\nabla_{t_2-\delta}}|^2 e^{-\frac{|x-x_0|^2}{t_0+\delta}} dV \\ &= \mathcal{F}_{x_0, t_0+\delta}(\nabla_{t_2-\delta}). \end{aligned}$$

We set $\delta = t_2 - t_1$ and observe that, combining (3.15) with (3.18),

$$\lambda(\nabla_{t_2}) - \epsilon \leq \mathcal{F}_{x_0, t_0}(\nabla_{t_2}) \leq \mathcal{F}_{x_0, t_0+t_2-t_1}(\nabla_{t_1}) \leq \lambda(\nabla_{t_1}).$$

Since this holds for each $\epsilon > 0$ we conclude the desired monotonicity. The result follows. \square

3.3. Second variation. For the computations of the following proposition refer to the note of §3.2 regarding the convention on variation parameter subscripts. Again, these variational calculations require the variation of the connection to be in a certain weighted Sobolev space which we suppress.

Proposition 3.9 (Second variation). *Let Γ_s , x_s and t_s be one parameter families of connections, points in \mathbb{R}^n , and positive real numbers respectively, and set*

$$\dot{t}_s := \frac{dt_s}{ds}, \quad \ddot{t}_s := \frac{d\dot{t}_s}{ds}, \quad \dot{x}_s := \frac{dx_s}{ds}, \quad \ddot{x}_s := \frac{d\dot{x}_s}{ds}, \quad \dot{\Gamma}_s := \frac{d\Gamma_s}{ds}, \quad \ddot{\Gamma}_s := \frac{d\dot{\Gamma}_s}{ds}.$$

Then

$$\begin{aligned} & \frac{d^2}{ds^2} [\mathcal{F}_{x_s, t_s}(\nabla_s)] \\ &= \int_{\mathbb{R}^n} \left(2(\ddot{t}_s t_s + \dot{t}_s^2) + t_s^2(\mathbf{g}_s^2 + \dot{\mathbf{g}}_s) + 4\dot{t}_s t_s \mathbf{g}_s \right) |F_s|^2 G_s dV \\ &+ \int_{\mathbb{R}^n} \left\langle (8\dot{t}_s t_s + 2t_s^2 \mathbf{g}_s) \dot{\Gamma}_s + 8t_s^2 \ddot{\Gamma}_s, S_{x_s, t_s}(\nabla_s) \right\rangle G_s dV \\ &+ 4t_s^2 \int_{\mathbb{R}^n} \left\langle \dot{\Gamma}_s, D_s^* D_s \dot{\Gamma}_s + \left(\frac{x - x_s}{2t_s} \right) \lrcorner D_s \dot{\Gamma}_s + [\dot{\Gamma}_s, F_s]^\# \right\rangle G_s dV \\ &- 4t_s \int_{\mathbb{R}^n} \left\langle \dot{\Gamma}_s, (\dot{t}_s(x - x_s) + \dot{x}_s) \lrcorner F_s \right\rangle G_s dV. \end{aligned}$$

where

$$(3.19) \quad \mathbf{g}_s := \left(-\frac{n\dot{t}_s}{2t_s} + \frac{\dot{t}_s |x - x_s|^2}{4t_s^2} + \frac{\langle \dot{x}_s, x - x_s \rangle}{2t_s} \right).$$

Proof. Prior to computing the main expression we perform some necessary side computations of differentiation. First observe that

$$\frac{\partial \mathbf{g}}{\partial x^i} = \left(\frac{\dot{t}_s}{2t_s} (x - x_s)^i + \frac{\dot{x}_s^i}{2t_s} \right).$$

We next take the second variation of \mathcal{F}_{x_s, t_s} and separate the resulting expression into labeled quantities:

$$\begin{aligned} (3.20) \quad & \frac{d^2}{ds^2} \left[\int_{\mathbb{R}^n} |F_s|^2 G_s dV \right] = \frac{d}{ds} \left(\int_{\mathbb{R}^n} \left(2\langle D_s \dot{\Gamma}_s, F_s \rangle + \mathbf{g}_s |F_s|^2 \right) G_s dV \right) \\ &= \int_{\mathbb{R}^n} \left(\underbrace{2\langle \partial_s (D_s \dot{\Gamma}_s), F_s \rangle}_{T_1} + \underbrace{2\langle D_s \dot{\Gamma}_s, D_s \dot{\Gamma}_s \rangle}_{T_2} + \underbrace{(\dot{\mathbf{g}}_s + \mathbf{g}_s^2) |F_s|^2}_{T_3} + \underbrace{4\mathbf{g}_s \langle D \dot{\Gamma}_s, F_s \rangle}_{T_4} \right) G_s dV. \end{aligned}$$

We first address the T_1 quantity. Since

$$\begin{aligned} \partial_s D_s \dot{\Gamma}_s &= \partial_s \left(\nabla_i \dot{\Gamma}_{j\alpha}^\beta \right) \\ &= \partial_t \left(\partial_i \dot{\Gamma}_{j\alpha}^\beta - \Gamma_{i\alpha}^\delta \dot{\Gamma}_{j\delta}^\beta + \Gamma_{i\delta}^\beta \dot{\Gamma}_{j\alpha}^\delta \right) \\ &= \partial_i \ddot{\Gamma}_{j\alpha}^\beta - \dot{\Gamma}_{i\alpha}^\delta \dot{\Gamma}_{j\delta}^\beta - \Gamma_{i\alpha}^\delta \ddot{\Gamma}_{j\delta}^\beta + \dot{\Gamma}_{i\delta}^\beta \dot{\Gamma}_{j\alpha}^\delta + \Gamma_{i\delta}^\beta \ddot{\Gamma}_{j\alpha}^\delta, \end{aligned}$$

it follows that

$$\begin{aligned}\partial_s \left(\nabla_i \dot{\Gamma}_{j\alpha}^\beta - \nabla_j \dot{\Gamma}_{i\alpha}^\beta \right) &= \partial_i \ddot{\Gamma}_{j\alpha}^\beta - \dot{\Gamma}_{i\alpha}^\delta \dot{\Gamma}_{j\delta}^\beta - \Gamma_{i\alpha}^\delta \ddot{\Gamma}_{j\delta}^\beta + \dot{\Gamma}_{i\delta}^\beta \dot{\Gamma}_{j\alpha}^\delta + \Gamma_{i\delta}^\beta \ddot{\Gamma}_{j\alpha}^\delta \\ &\quad - \partial_j \ddot{\Gamma}_{i\alpha}^\beta + \dot{\Gamma}_{j\alpha}^\delta \dot{\Gamma}_{i\delta}^\beta + \Gamma_{j\alpha}^\delta \ddot{\Gamma}_{i\delta}^\beta - \dot{\Gamma}_{j\delta}^\beta \dot{\Gamma}_{i\alpha}^\delta - \Gamma_{j\delta}^\beta \ddot{\Gamma}_{i\alpha}^\delta \\ &= D_i \ddot{\Gamma}_{j\alpha}^\beta - 2 \dot{\Gamma}_{i\alpha}^\delta \dot{\Gamma}_{j\delta}^\beta + 2 \dot{\Gamma}_{i\delta}^\beta \dot{\Gamma}_{j\alpha}^\delta.\end{aligned}$$

Now applying this to the expression above,

$$\begin{aligned}\int_{\mathbb{R}^n} \langle \partial_s (D_s \dot{\Gamma}_s), F_s \rangle G_s dV_g &= - \int_{\mathbb{R}^n} \left(\nabla_i \ddot{\Gamma}_{j\alpha}^\beta - \nabla_j \ddot{\Gamma}_{i\alpha}^\beta \right) F_{ij\beta}^\alpha G dV - 2 \int_{\mathbb{R}^n} \left(\dot{\Gamma}_{i\alpha}^\delta \dot{\Gamma}_{j\delta}^\beta - \dot{\Gamma}_{i\delta}^\beta \dot{\Gamma}_{j\alpha}^\delta \right) F_{ij\beta}^\alpha G dV \\ &= 4 \int_{\mathbb{R}^n} \ddot{\Gamma}_{j\alpha}^\beta \left(\nabla_i F_{ij\beta}^\alpha \right) G dV + 4 \int_{\mathbb{R}^n} \left(\ddot{\Gamma}_{j\alpha}^\beta \right) F_{ij\beta}^\alpha \left(\nabla_i G \right) dV \\ &\quad - 2 \int_{\mathbb{R}^n} \left(\dot{\Gamma}_{i\alpha}^\delta \dot{\Gamma}_{j\delta}^\beta F_{ij\beta}^\alpha - \dot{\Gamma}_{i\delta}^\beta F_{ij\beta}^\alpha \dot{\Gamma}_{j\alpha}^\delta \right) G dV \\ &= 4 \int_{\mathbb{R}^n} \ddot{\Gamma}_{j\alpha}^\beta \left(\nabla_i F_{ij\beta}^\alpha \right) G dV - 4 \int_{\mathbb{R}^n} \ddot{\Gamma}_{j\alpha}^\beta F_{ij\beta}^\alpha \frac{(x-x_s)^i}{2t_s} G dV \\ &\quad - 2 \int_{\mathbb{R}^n} \left(\dot{\Gamma}_{i\alpha}^\delta \dot{\Gamma}_{j\delta}^\beta F_{ij\beta}^\alpha - \dot{\Gamma}_{i\delta}^\beta F_{ij\beta}^\alpha \dot{\Gamma}_{j\alpha}^\delta \right) G dV.\end{aligned}$$

Therefore in coordinate invariant form,

$$(3.21) \quad \int_{\mathbb{R}^n} T_1 G_s dV = -8 \int_{\mathbb{R}^n} \langle \ddot{\Gamma}_s, S_{x_s, t_s} \rangle G_s dV + 4 \int_{\mathbb{R}^n} \langle \dot{\Gamma}_s, [\dot{\Gamma}_s, F_s]^\# \rangle G_s dV.$$

Next for the T_2 quantity we expand terms and then applying integration by parts:

$$\begin{aligned}(3.22) \quad \int_{\mathbb{R}^n} T_2 G_s dV &= -2 \int_{\mathbb{R}^n} D_i \dot{\Gamma}_{j\alpha}^\beta D_i \dot{\Gamma}_{j\beta}^\alpha G dV \\ &= -2 \int_{\mathbb{R}^n} (\nabla_i \dot{\Gamma}_{j\alpha}^\beta - \nabla_j \dot{\Gamma}_{i\alpha}^\beta) D_i \dot{\Gamma}_{j\beta}^\alpha G dV \\ &= -2 \int_{\mathbb{R}^n} \left(\nabla_i \dot{\Gamma}_{j\alpha}^\beta D_i \dot{\Gamma}_{j\beta}^\alpha \right) G dV + 2 \int_{\mathbb{R}^n} \left(\nabla_j \dot{\Gamma}_{i\alpha}^\beta D_i \dot{\Gamma}_{j\beta}^\alpha \right) G dV \\ &= -2 \int_{\mathbb{R}^n} \left(\nabla_i \dot{\Gamma}_{j\alpha}^\beta D_i \dot{\Gamma}_{j\beta}^\alpha \right) G dV + 2 \int_{\mathbb{R}^n} \left(\nabla_i \dot{\Gamma}_{j\alpha}^\beta D_j \dot{\Gamma}_{i\beta}^\alpha \right) G dV \\ &= 2 \int_{\mathbb{R}^n} \dot{\Gamma}_{j\alpha}^\beta \left(\nabla_i D_i \dot{\Gamma}_{j\beta}^\alpha - \left(\frac{x-x_s}{2t_s} \right)^i D_i \dot{\Gamma}_{j\beta}^\alpha \right) G dV \\ &\quad - 2 \int_{\mathbb{R}^n} \dot{\Gamma}_{j\alpha}^\beta \left(\nabla_i D_j \dot{\Gamma}_{i\beta}^\alpha - \left(\frac{x-x_s}{2t_s} \right)^i D_j \dot{\Gamma}_{i\beta}^\alpha \right) G dV \\ &= 2 \int_{\mathbb{R}^n} \dot{\Gamma}_{j\alpha}^\beta \left(\nabla_i \left(D_i \dot{\Gamma}_{j\beta}^\alpha - D_j \dot{\Gamma}_{i\beta}^\alpha \right) + \left(\left(\frac{x-x_s}{2t_s} \right)^i D_j \dot{\Gamma}_{i\beta}^\alpha - \left(\frac{x-x_s}{2t_s} \right)^i D_i \dot{\Gamma}_{j\beta}^\alpha \right) \right) G dV \\ &= 2 \int_{\mathbb{R}^n} \dot{\Gamma}_{j\alpha}^\beta \left(\nabla_i D_i \dot{\Gamma}_{j\beta}^\alpha - \left(\frac{x-x_s}{t_s} \right)^i (D_i \dot{\Gamma}_{j\beta}^\alpha) \right) G dV \\ &= 4 \int_{\mathbb{R}^n} \left\langle \dot{\Gamma}_s, D_s^* D_s \dot{\Gamma}_s + \left(\frac{x-x_s}{2t_s} \right) \lrcorner D_s \dot{\Gamma}_s \right\rangle G_s dV.\end{aligned}$$

Lastly we expand the T_4 quantity by expanding and integrating by parts:

$$\begin{aligned}
\int_{\mathbb{R}^n} T_4 G_s dV &= 4 \int_{\mathbb{R}^n} \left(\mathbf{g}_s \langle D_s \dot{\Gamma}_s, F_s \rangle \right) G_s dV \\
&= -4 \int_{\mathbb{R}^n} \left(\mathbf{g}(D_i \dot{\Gamma}_{j\alpha}^\beta) F_{ij\beta}^\alpha \right) G dV \\
&= -8 \int_{\mathbb{R}^n} \left(\mathbf{g}(\nabla_i \dot{\Gamma}_{j\alpha}^\beta) F_{ij\beta}^\alpha \right) G dV \\
(3.23) \quad &= 8 \int_{\mathbb{R}^n} \left((\partial_i \mathbf{g}) \frac{(x-x_s)^i}{2t_s} \right) \dot{\Gamma}_{j\alpha}^\beta F_{ij\beta}^\alpha G dV + 8 \int_{\mathbb{R}^n} \mathbf{g} \dot{\Gamma}_{j\alpha}^\beta (\nabla_i F_{ij\beta}^\alpha) G dV \\
&= 8 \int_{\mathbb{R}^n} \left\langle \dot{\Gamma}_s, \mathbf{g}_s \left(D_s^* F_s + \left(\frac{x-x_s}{2t_s} \lrcorner F_s \right) - (\partial \mathbf{g}_s) \lrcorner F_s \right) \right\rangle G_s dV \\
&= 8 \int_{\mathbb{R}^n} \left\langle \dot{\Gamma}_s, \mathbf{g}_s (S_{x_s, t_s} - (\partial \mathbf{g}_s) \lrcorner F_s) \right\rangle G_s dV.
\end{aligned}$$

Combining all quantities (3.21), (3.22), and (3.23) together into (3.20) yields

$$\begin{aligned}
(3.24) \quad \frac{d^2}{ds^2} \left[\int_{\mathbb{R}^n} |F_s|^2 G_s dV \right] &= 8 \int_{\mathbb{R}^n} \langle \ddot{\Gamma}_s, S_{x_s, t_s} \rangle G_s dV + 2 \int_{\mathbb{R}^n} \langle \dot{\Gamma}_s, \mathbf{g}_s S_{x_s, t_s} \rangle G_s dV + \int_{\mathbb{R}^n} (\dot{\mathbf{g}}_s + \mathbf{g}_s^2) |F_s|^2 G_s dV \\
&\quad + 4 \int_{\mathbb{R}^n} \left\langle \dot{\Gamma}_s, D_s^* D_s \dot{\Gamma}_s + \left(\frac{x-x_s}{2t_s} \right) \lrcorner D_s \dot{\Gamma}_s + [\dot{\Gamma}_s, F_s]^\# \right\rangle G_s dV \\
&\quad - 4 \int_{\mathbb{R}^n} \left\langle \dot{\Gamma}_s, \left(\frac{\dot{t}_s}{t_s} (x - x_s) + \frac{\dot{x}_s}{t_s} \right) \lrcorner F_s \right\rangle G_s dV.
\end{aligned}$$

Now we incorporate the temporal parameter and differentiate the quantity

$$(3.25) \quad \frac{d^2}{ds^2} [\mathcal{F}_{x_s, t_s}] = \frac{\partial}{\partial s} \left[2\dot{t}_s t_s \mathcal{F}_{x_s, t_s} + t_s^2 \frac{\partial}{\partial s} [\mathcal{F}_{x_s, t_s}] \right] = 2(\ddot{t}_s t_s + \dot{t}_s^2) \mathcal{F}_{x_s, t_s} + 4\dot{t}_s t_s \frac{\partial}{\partial s} [\mathcal{F}_{x_s, t_s}] + t_s^2 \frac{\partial^2}{\partial s^2} [\mathcal{F}_{x_s, t_s}].$$

We insert the terms from (3.24) and (3.10) into (3.25) and obtain

$$\begin{aligned}
\frac{d^2}{ds^2} [\mathcal{F}_{x_s, t_s}] &= 2(\ddot{t}_s t_s + \dot{t}_s^2) \int_{\mathbb{R}^n} |F_s|^2 G_s dV + 4\dot{t}_s t_s \int_{\mathbb{R}^n} |F_s|^2 \mathbf{g}_s G_s dV + 8\dot{t}_s t_s \int_{\mathbb{R}^n} \langle \dot{\Gamma}_s, S_{x_s, t_s}(\nabla_s) \rangle G_s dV \\
&\quad + 8\dot{t}_s^2 \int_{\mathbb{R}^n} \langle \ddot{\Gamma}_s, S_{x_s, t_s}(\nabla_s) \rangle G_s dV + 2\dot{t}_s^2 \int_{\mathbb{R}^n} \langle \dot{\Gamma}_s, \mathbf{g}_s S_{x_s, t_s}(\nabla_s) \rangle G_s dV + t_s^2 \int_{\mathbb{R}^n} (\dot{\mathbf{g}}_s + \mathbf{g}_s^2) |F_s|^2 G_s dV \\
&\quad + 4\dot{t}_s^2 \int_{\mathbb{R}^n} \left\langle \dot{\Gamma}_s, D_s^* D_s \dot{\Gamma}_s + \left(\frac{x-x_s}{2t_s} \right) \lrcorner D_s \dot{\Gamma}_s + [\dot{\Gamma}_s, F_s]^\# \right\rangle G_s dV \\
&\quad - 4\dot{t}_s^2 \int_{\mathbb{R}^n} \left\langle \dot{\Gamma}_s, \left(\frac{\dot{t}_s}{t_s} (x - x_s) + \frac{\dot{x}_s}{t_s} \right) \lrcorner F_s \right\rangle G_s dV \\
&= \int_{\mathbb{R}^n} (2(\ddot{t}_s t_s + \dot{t}_s^2) + t_s^2 (\mathbf{g}_s^2 + \dot{\mathbf{g}}_s) + 4\dot{t}_s t_s \mathbf{g}_s) |F_s|^2 G_s dV \\
&\quad + \int_{\mathbb{R}^n} \langle \dot{\Gamma}_s, (8\dot{t}_s t_s + 2\dot{t}_s^2 \mathbf{g}_s) S_{x_s, t_s}(\nabla_s) \rangle G_s dV + 8\dot{t}_s^2 \int_{\mathbb{R}^n} \langle \ddot{\Gamma}_s, S_{x_s, t_s}(\nabla_s) \rangle G_s dV \\
&\quad + 4\dot{t}_s^2 \int_{\mathbb{R}^n} \left\langle \dot{\Gamma}_s, D_s^* D_s \dot{\Gamma}_s + \left(\frac{x-x_s}{2t_s} \right) \lrcorner D_s \dot{\Gamma}_s + [\dot{\Gamma}_s, F_s]^\# \right\rangle G_s dV \\
&\quad - 4\dot{t}_s^2 \int_{\mathbb{R}^n} \left\langle \dot{\Gamma}_s, \left(\frac{\dot{t}_s}{t_s} (x - x_s) + \frac{\dot{x}_s}{t_s} \right) \lrcorner F_s \right\rangle G_s dV.
\end{aligned}$$

The result follows. \square

We now specialize this result to the case of solitons. For notational clarity, after evaluating at $s = 0$, we excise the subscript except for those of the base point (x_0, t_0) and the heat kernel.

Corollary 3.10 (Second Variation for Shrinkers). *Suppose that ∇ is an (x_0, t_0) -soliton. Then*

$$\begin{aligned} \mathcal{F}''_{x_0, t_0}(\dot{t}, \dot{x}, \dot{\nabla}) &= \frac{d^2}{ds^2} [\mathcal{F}_{x_s, t_s}(\nabla_s)] \Big|_{s=0} \\ &= -4t_0 \dot{t}_0^2 \int_{\mathbb{R}^n} |D^* F|^2 G_0 dV - 2t \int_{\mathbb{R}^n} |F \lrcorner \dot{x}_0|^2 G_0 dV \\ &\quad + 4t^2 \int_{\mathbb{R}^n} \left\langle \dot{\Gamma}, D^* D \dot{\Gamma} + \left(\frac{x - x_0}{2t_0} \right) \lrcorner D \dot{\Gamma} + [\dot{\Gamma}, F]^\# \right\rangle G_0 dV \\ &\quad - 4t_0 \int_{\mathbb{R}^n} \left\langle \dot{\Gamma}, (\dot{t}_0(x - x_0) + \dot{x}_0) \lrcorner F \right\rangle G_0 dV. \end{aligned}$$

Proof. We first compute to identities concerning \mathbf{g}_s to be used for the following argument. First we have

$$(3.26) \quad \frac{\partial \mathbf{g}}{\partial s} = -\frac{n}{2} \left(\frac{\ddot{t}_s}{t_s} - \frac{\dot{t}_s^2}{t_s^2} \right) + \frac{|x - x_s|^2}{4} \left(\frac{\ddot{t}_s}{t_s^2} - \frac{2\dot{t}_s^2}{t_s^3} \right) - \frac{\dot{t}_s \langle \dot{x}_s, x - x_s \rangle}{t_s^2} + \frac{\langle \ddot{x}_s, x - x_s \rangle}{2t_s} - \frac{|\dot{x}_s|^2}{2t_s}.$$

Then

$$\begin{aligned} (3.27) \quad \mathbf{g}_s^2 &= \left(-\frac{n\dot{t}_s}{2t_s} + \frac{\dot{t}_s |x - x_s|^2}{4t_s^2} + \frac{\langle \dot{x}_s, x - x_s \rangle}{2t_s} \right)^2 \\ &= \frac{n^2 \dot{t}_s^2}{4t_s^2} - \frac{n\dot{t}_s^2 |x - x_s|^2}{4t_s^3} - \frac{n\dot{t}_s \langle \dot{x}_s, x - x_s \rangle}{2t_s^2} + \frac{\dot{t}_s |x - x_s|^2 \langle \dot{x}_s, x - x_s \rangle}{4t_s^3} + \frac{\dot{t}_s^2 |x - x_s|^4}{16t_s^4} + \frac{\langle \dot{x}_s, x - x_s \rangle^2}{4t_s^2}. \end{aligned}$$

Adding the two quantities (3.26) and (3.27) and labeling with the corresponding items of Corollary 3.4 we obtain

$$\begin{aligned} \dot{\mathbf{g}} + \mathbf{g}^2 &= \left(-\frac{n\dot{t}_0}{2t_0} + \frac{n\dot{t}_0^2}{2t_0^2} + \frac{n^2\dot{t}_0^2}{4t_0^2} \right) + \left(\frac{|x - x_0|^2}{4t_0} \left(\frac{\ddot{t}_0}{t_0} - \frac{n\dot{t}_0^2}{t_0^2} - \frac{2\dot{t}_0^2}{t_0^2} \right) \right)_a + \left(\langle \dot{x}_0, x - x_0 \rangle \left(-\frac{\dot{t}_0}{t_0^2} - \frac{n\dot{t}_0}{2t_0^2} \right) \right)_b \\ &\quad + \left(|x - x_0|^2 \langle \dot{x}_0, x - x_0 \rangle \left(\frac{\dot{t}_0}{4t_0^3} \right) \right)_d + \left(\langle \ddot{x}_0, x - x_0 \rangle \frac{1}{2t_0} \right)_b + \left(|\dot{x}_0|^2 \left(\frac{-1}{2t_0} \right) \right) \\ &\quad + \left(|x - x_0|^4 \left(\frac{\dot{t}_0^2}{16t_0^4} \right) \right)_c + \left(\langle \dot{x}_0, x - x_0 \rangle^2 \left(\frac{1}{4t_0^2} \right) \right)_e. \end{aligned}$$

Applying the said identities of Corollary 3.4 we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} (\dot{\mathbf{g}} + \mathbf{g}^2) |F|^2 G_s dV &= \int_{\mathbb{R}^n} \left(\left(-\frac{n\ddot{t}_0}{2t_0} + \frac{nt_0^2}{2t_0^2} + \frac{n^2t_0^2}{4t_0^2} \right) + \left(\frac{n-4}{2} \left(\frac{\ddot{t}_0}{t_0} - \frac{nt_0^2}{t_0^2} - \frac{2t_0^2}{t_0^2} \right) \right) \right) |F|^2 G_0 dV \\
&\quad + \int_{\mathbb{R}^n} \left(\left(|\dot{x}_0|^2 \left(\frac{-1}{2t_0} \right) \right) + 4(n-2)(n-4)t_0^2 \left(\frac{t_0^2}{16t_0^4} \right) + \left(\frac{1}{2t_0} \right) |\dot{x}_0|^2 \right) |F|^2 G_0 dV \\
&\quad - 4 \left(\frac{t_0^2}{t_0} \right) \int_{\mathbb{R}^n} |D^* F|^2 G_0 dV - 8t_0 \left(\frac{1}{4t_0^2} \right) \int_{\mathbb{R}^n} |F \lrcorner \dot{x}_0|^2 G_0 dV \\
&= \int_{\mathbb{R}^n} \left(\left(-\frac{n\ddot{t}_0}{2t_0} + \frac{nt_0^2}{2t_0^2} + \frac{n^2t_0^2}{4t_0^2} \right) + \left(\frac{n-4}{2} \left(\frac{\ddot{t}_0}{t_0} - \frac{nt_0^2}{t_0^2} - \frac{2t_0^2}{t_0^2} \right) \right) \right) |F_s|^2 G_0 dV \\
&\quad + \int_{\mathbb{R}^n} \left((n-2)(n-4) \left(\frac{t_0^2}{4t_0^2} \right) \right) |F|^2 G_0 dV \\
&\quad - 4 \left(\frac{t_0^2}{t_0} \right) \int_{\mathbb{R}^n} |D^* F|^2 G_0 dV - 8t_0 \left(\frac{1}{4t_0^2} \right) \int_{\mathbb{R}^n} |F \lrcorner \dot{x}_0|^2 G_0 dV.
\end{aligned}$$

We collect and simplify the coefficients of the integrands multiplied against $|F|^2 G$ of the first and second line to obtain

$$\begin{aligned}
&\left(\frac{4t_0^2}{t_0^2} - \frac{2\ddot{t}_0}{t_0} + \frac{2t_0^2}{t_0^2} \right) + n \left(-\frac{\ddot{t}_0}{2t_0} + \frac{t_0^2}{2t_0^2} + \frac{2t_0^2}{t_0^2} + \frac{\ddot{t}_0}{2t_0} - \frac{t_0^2}{t_0^2} - \frac{3t_0^2}{2t_0^2} \right) + n^2 \left(\frac{t_0^2}{4t_0^2} - \frac{t_0^2}{2t_0^2} + \frac{t_0^2}{4t_0^2} \right) \\
&= \frac{6t_0^2}{t_0^2} - \frac{2\ddot{t}_0}{t_0}.
\end{aligned}$$

Therefore we conclude that

$$t_0^2 \int_{\mathbb{R}^n} (\dot{\mathbf{g}} + \mathbf{g}^2) |F|^2 G dV = \int_{\mathbb{R}^n} (6t_0^2 - 2\ddot{t}_0 t_0) |F|^2 G dV - 4t_0 t_0^2 \int_{\mathbb{R}^n} |D^* F|^2 G dV - 2t_0 \int_{\mathbb{R}^n} |F \lrcorner \dot{x}_0|^2 G dV.$$

We combine terms, apply Corollary (3.4) (b) and then (a),

$$\begin{aligned}
&\int_{\mathbb{R}^n} (2(\ddot{t}_0 t_0 + t_0^2) + t_0^2(\mathbf{g}^2 + \dot{\mathbf{g}}) + 4t_0 t_0 \mathbf{g}) |F|^2 G_0 dV \\
&= \int_{\mathbb{R}^n} \left(2t_0^2 - 2nt_0^2 + \frac{t_0^2 |x - x_0|^2}{t_0} + t_0 \langle \dot{x}_0, x - x_0 \rangle \right) |F|^2 G_0 dV + \int_{\mathbb{R}^n} 6t_0^2 |F|^2 G_0 dV \\
&\quad - 4t_0 t_0^2 \int_{\mathbb{R}^n} |D^* F|^2 G_0 dV - 2t_0 \int_{\mathbb{R}^n} |F \lrcorner \dot{x}_0|^2 G_0 dV \\
&= -4t_0 t_0^2 \int_{\mathbb{R}^n} |D^* F|^2 G_0 dV - 2t_0 \int_{\mathbb{R}^n} |F \lrcorner \dot{x}_0|^2 G_0 dV.
\end{aligned}$$

Therefore we conclude that

$$\begin{aligned}
\frac{d^2}{ds^2} [\mathcal{F}_{x_s, t_s}(\nabla_s)] \Big|_{s=0} &= -4t_0 t_0^2 \int_{\mathbb{R}^n} |D^* F|^2 G_0 dV - 2t_0 \int_{\mathbb{R}^n} |F \lrcorner \dot{x}_0|^2 G_0 dV \\
&\quad + 4t_0^2 \int_{\mathbb{R}^n} \left\langle \dot{\Gamma}, D^* D \dot{\Gamma} + \left(\frac{x - x_0}{2t_0} \right) \lrcorner D \dot{\Gamma} + [\dot{\Gamma}, F]^\# \right\rangle G_0 dV \\
&\quad - 4t_0^2 \int_{\mathbb{R}^n} \left\langle \dot{\Gamma}, (t_0(x - x_0) + \dot{x}_0) \lrcorner F \right\rangle G_0 dV.
\end{aligned}$$

The result follows. \square

4. \mathcal{F} -STABILITY AND GAP THEOREM

In this section we establish a criterion for checking entropy stability (Theorem 4.5) and also prove a gap theorem for shrinkers (Theorem 1.1). The basic observation behind the stability condition is that for a shrinker, the second variation operator L *always* has negative eigenvalues, one corresponding to the Yang-Mills flow direction itself, which is the same as moving in time while scaling, the other corresponding to translation in space. This in some sense is the whole reason for explicitly including the space and time parameters in the definition of entropy, as we then show in Theorem 4.5 that these directions can be accounted for by an appropriate choice of variation in the basepoint.

Recalling Corollary 3.10 we define the operator L_{x_0, t_0} by

$$L_{x_0, t_0} : \Lambda^1(\text{End } E) \rightarrow \Lambda^1(\text{End } E) : B \mapsto D^*DB + \left(\frac{x - x_0}{2t_0} \right) \lrcorner DB + [B, F]^\#.$$

In particular, we set $L := L_{0,1}$. By the Bochner formula this is equal to

$$L_{x_0, t_0}(B) = -\Delta B - \nabla D^*B - 2[B, F]^\#.$$

We ultimately only want to apply L to elements of an appropriate weighted Sobolev space. For a given $\nabla \in \mathfrak{S}$ set

$$(4.1) \quad W_{\nabla}^{2,2} := \left\{ B \in \Lambda^1(\text{End } E) : \int_{\mathbb{R}^n} (|B|^2 + |\nabla B|^2 + |LB|^2) e^{\frac{-|x|^2}{4}} dV < \infty \right\}.$$

4.1. Second variation operator.

Definition 4.1. A soliton ∇ is called \mathcal{F} -stable if for any $B \in W_{\nabla}^{2,2}$ there exist a real number σ and a constant vector field V such that $\mathcal{F}_{0,1}''(q, V, B) \geq 0$.

Lemma 4.2. *Let $V = V^i \partial_i$ be some vector field. Then*

$$(4.2) \quad \begin{aligned} L(V \lrcorner F)_{k\alpha}^\beta &= (\nabla_p \nabla_k V^i) F_{ip\alpha}^\beta - (\nabla_p \nabla_p V^i) F_{ik\alpha}^\beta - \frac{V^i}{2} F_{ik\alpha}^\beta \\ &\quad - (\nabla_p V^i) \left((\nabla_p F_{ik\alpha}^\beta) + (\nabla_i F_{pk\alpha}^\beta) - \frac{x_p}{2} F_{ik\alpha}^\beta \right). \end{aligned}$$

Proof. We compute the first term of the L operator and obtain

$$(4.3) \quad \begin{aligned} D^*D(V \lrcorner F)_{k\alpha}^\beta &= -\nabla_p D_p(V \lrcorner F) \\ &= -\nabla_p \nabla_p \left(V^i F_{ik\alpha}^\beta \right) + \nabla_p \nabla_k \left(V^i F_{ip\alpha}^\beta \right) \\ &= -\nabla_p \left((\nabla_p V^i) F_{ik\alpha}^\beta + V^i (\nabla_p F_{ik\alpha}^\beta) - (\nabla_k V^i) F_{ip\alpha}^\beta - V^i (\nabla_k F_{ip\alpha}^\beta) \right) \\ &= (\nabla_p \nabla_k V^i) F_{ip\alpha}^\beta - (\nabla_p \nabla_p V^i) F_{ik\alpha}^\beta + \left[V^i (\nabla_p \nabla_k F_{ip\alpha}^\beta) - V^i (\nabla_p \nabla_p F_{ik\alpha}^\beta) \right]_T \\ &\quad - 2(\nabla_p V^i) (\nabla_p F_{ik\alpha}^\beta) + (\nabla_p V^i) (\nabla_k F_{ip\alpha}^\beta) + (\nabla_k V^i) (\nabla_p F_{ip\alpha}^\beta). \end{aligned}$$

Now we manipulate T . Observe that

$$\begin{aligned}
T &= -V^i \left(\nabla_p \nabla_p F_{ik\alpha}^\beta - \nabla_p \nabla_k F_{ip\alpha}^\beta \right) \\
&= -V^i \left(-\nabla_p \left(\nabla_k F_{pi\alpha}^\beta + \nabla_i F_{kp\alpha}^\beta \right) - \nabla_p \nabla_k F_{ip\alpha}^\beta \right) \\
&= V^i \left(\nabla_p \nabla_i F_{kp\alpha}^\beta \right) \\
&= -V^i (\nabla_i \nabla_p F_{pk\alpha}^\beta) + V_i [\nabla_p, \nabla_i] F_{kp\alpha}^\beta \\
&= -V^i (\nabla_i \nabla_p F_{pk\alpha}^\beta) + V_i (F_{pi\delta}^\beta F_{kp\alpha}^\delta + F_{pi\alpha}^\delta F_{kp\delta}^\beta) \\
&= V^i (\nabla_i (D^* F)_{k\alpha}^\beta) - V_i (F_{ip\delta}^\beta F_{kp\alpha}^\delta - F_{ip\alpha}^\delta F_{kp\delta}^\beta) \\
&= -V^i \nabla_i \left(\frac{x_p}{2} F_{pk\alpha}^\beta \right) + ([F, (V \lrcorner F)]^\#)_{k\alpha}^\beta \\
&= -\frac{V^i}{2} \left(F_{ik\alpha}^\beta - x_p \nabla_i F_{pk\alpha}^\beta \right) + ([F, (V \lrcorner F)]^\#)_{k\alpha}^\beta \\
&= -\frac{V^i}{2} F_{ik\alpha}^\beta + \frac{V_i}{2} \left(x_p (\nabla_k F_{ip\alpha}^\beta + \nabla_p F_{ki\alpha}^\beta) \right) + ([F, (V \lrcorner F)]^\#)_{k\alpha}^\beta \\
&= -\frac{V^i}{2} F_{ik\alpha}^\beta + \frac{V_i}{2} x_p \nabla_k F_{ip\alpha}^\beta - \frac{V_i}{2} x_p \nabla_p F_{ik\alpha}^\beta + ([F, (V \lrcorner F)]^\#)_{k\alpha}^\beta.
\end{aligned}$$

Therefore we conclude that, applying the identity of T to (4.3),

$$\begin{aligned}
(4.4) \quad D^* D [V \lrcorner F]_{k\alpha}^\beta &= (\nabla_p \nabla_k V^i) F_{ip\alpha}^\beta - (\nabla_p \nabla_p V^i) F_{ik\alpha}^\beta \\
&\quad - 2(\nabla_p V^i)(\nabla_p F_{ik\alpha}^\beta) + (\nabla_p V^i)(\nabla_k F_{ip\alpha}^\beta) + (\nabla_k V^i)(\nabla_p F_{ip\alpha}^\beta) \\
&\quad - \frac{V^i}{2} F_{ik\alpha}^\beta + \frac{V_i}{2} x_p \nabla_k F_{ip\alpha}^\beta - \frac{V_i}{2} x_p \nabla_p F_{ik\alpha}^\beta + ([F, (V \lrcorner F)]^\#)_{k\alpha}^\beta.
\end{aligned}$$

Next we compute

$$\begin{aligned}
(4.5) \quad \frac{x}{2} \lrcorner D (V \lrcorner F) &= \frac{x_p}{2} D_p \left(V^i F_{ik\alpha}^\beta \right) \\
&= \frac{x_p}{2} \left(\nabla_p \left(V^i F_{ik\alpha}^\beta \right) - \nabla_k \left(V^i F_{ip\alpha}^\beta \right) \right) \\
&= \frac{x_p}{2} \left((\nabla_p V^i) F_{ik\alpha}^\beta + V^i (\nabla_p F_{ik\alpha}^\beta) - (\nabla_k V^i) F_{ip\alpha}^\beta - V^i (\nabla_k F_{ip\alpha}^\beta) \right).
\end{aligned}$$

Therefore we have that, applying (4.4) and (4.5) to the formula for $L(V \lrcorner F)$, we have

$$\begin{aligned}
L(V \lrcorner F)_{k\alpha}^\beta &= (D^* D(V \lrcorner F))_{k\alpha}^\beta + \frac{x_p}{2} \lrcorner (D(V \lrcorner F))_{k\alpha}^\beta + \left([V \lrcorner F, F]^\# \right)_{k\alpha}^\beta \\
&= (\nabla_p \nabla_k V^i) F_{ip\alpha}^\beta - (\nabla_p \nabla_p V^i) F_{ik\alpha}^\beta \\
&\quad - 2(\nabla_p V^i)(\nabla_p F_{ik\alpha}^\beta) + (\nabla_p V^i)(\nabla_k F_{ip\alpha}^\beta) + (\nabla_k V^i)(\nabla_p F_{ip\alpha}^\beta) \\
&\quad - \frac{V^i}{2} F_{ik\alpha}^\beta + \frac{x_p}{2} (\nabla_p V^i) F_{ik\alpha}^\beta - \frac{x_p}{2} (\nabla_k V^i) F_{ip\alpha}^\beta \\
&= (\nabla_p \nabla_k V^i) F_{ip\alpha}^\beta - (\nabla_p \nabla_p V^i) F_{ik\alpha}^\beta \\
&\quad - 2(\nabla_p V^i)(\nabla_p F_{ik\alpha}^\beta) + (\nabla_p V^i)(\nabla_k F_{ip\alpha}^\beta) + (\nabla_k V^i)(DF)_{i\alpha}^\beta \\
&\quad - \frac{V^i}{2} F_{ik\alpha}^\beta + \frac{x_p}{2} (\nabla_p V^i) F_{ik\alpha}^\beta + \frac{x_p}{2} (\nabla_k V^i) F_{pi\alpha}^\beta \\
&= (\nabla_p \nabla_k V^i) F_{ip\alpha}^\beta - (\nabla_p \nabla_p V^i) F_{ik\alpha}^\beta \\
&\quad - (\nabla_p V^i)(\nabla_p F_{ik\alpha}^\beta) + (\nabla_p V^i) \left((\nabla_p F_{ki\alpha}^\beta) + (\nabla_k F_{ip\alpha}^\beta) \right) \\
&\quad - \frac{V^i}{2} F_{ik\alpha}^\beta + \frac{x_p}{2} (\nabla_p V^i) F_{ik\alpha}^\beta \\
&= (\nabla_p \nabla_k V^i) F_{ip\alpha}^\beta - (\nabla_p \nabla_p V^i) F_{ik\alpha}^\beta - \frac{V^i}{2} F_{ik\alpha}^\beta \\
&\quad - (\nabla_p V^i) \left((\nabla_p F_{ik\alpha}^\beta) + (\nabla_i F_{pk\alpha}^\beta) - \frac{x_p}{2} F_{ik\alpha}^\beta \right).
\end{aligned}$$

The result follows. □

Lemma 4.3 (Eigenforms of L). *For $\nabla \in \mathfrak{S}$, and a constant vector field V ,*

$$(4.6) \quad L(V \lrcorner F) = -\frac{1}{2} (V \lrcorner F).$$

Furthermore

$$(4.7) \quad L(D^* F) = -D^* F,$$

Proof. The identity (4.6) is an immediate corollary of Lemma 4.2, by simply evaluating (4.2) on a constant vector field $V = V^i \partial_i$. For the identity (4.7), we compute the following, applying the

first Bianchi identity to obtain

$$\begin{aligned}
(D^* D(D^* F))_{r\alpha}^\beta &= - (D^* D(\tfrac{x}{2} \lrcorner F))_{r\alpha}^\beta \\
&= \nabla_i \left(D_i \left(\tfrac{x_p}{2} F_{pr\alpha}^\beta \right) \right) \\
&= \tfrac{1}{2} \nabla_i \left(\nabla_i \left(x_p F_{pr\alpha}^\beta \right) - \nabla_r \left(x_p F_{pi\alpha}^\beta \right) \right) \\
&= \tfrac{1}{2} \nabla_i \left(F_{ir\alpha}^\beta + x_p \nabla_i F_{pr\alpha}^\beta - F_{ri\alpha}^\beta - x_p \nabla_r F_{pi\alpha}^\beta \right) \\
&= \tfrac{1}{2} \nabla_i \left(F_{ir\alpha}^\beta - F_{ri\alpha}^\beta \right) + \tfrac{1}{2} \nabla_i \left(x_p \nabla_i F_{pr\alpha}^\beta - x_p \nabla_r F_{pi\alpha}^\beta \right) \\
&= -(D^* F)_{r\alpha}^\beta + \tfrac{1}{2} \nabla_i \left(x_p \nabla_i F_{pr\alpha}^\beta + x_p (\nabla_i F_{rp\alpha}^\beta + \nabla_p F_{ir\alpha}^\beta) \right) \\
&= -(D^* F)_{r\alpha}^\beta + \tfrac{1}{2} \nabla_i \left(x_i \nabla_p F_{ir\alpha}^\beta \right) \\
&= -(D^* F)_{r\alpha}^\beta + \tfrac{1}{2} \left(\delta_{ip} \nabla_p F_{ir\alpha}^\beta + x_p \nabla_i \nabla_p F_{ir\alpha}^\beta \right) \\
&= -(D^* F)_{r\alpha}^\beta + \tfrac{1}{2} \left(\nabla_i F_{ir\alpha}^\beta + \tfrac{x_p}{2} \nabla_i \nabla_p F_{ir\alpha}^\beta \right) \\
&= -(D^* F)_{r\alpha}^\beta - \tfrac{1}{2} D^* F_{r\alpha}^\beta + \tfrac{x_p}{2} \nabla_p \nabla_i F_{ir\alpha}^\beta + \tfrac{x_p}{2} [\nabla_i, \nabla_p] F_{ir\alpha}^\beta \\
&= -\tfrac{3}{2} (D^* F)_{r\alpha}^\beta + \tfrac{x_p}{2} D_p \nabla_i F_{ir\alpha}^\beta + \tfrac{x_p}{2} \nabla_r \nabla_i F_{ip\alpha}^\beta + \tfrac{x_p}{2} \left(F_{ip\delta}^\beta F_{ir\alpha}^\delta - F_{ip\alpha}^\delta F_{ir\delta}^\beta \right) \\
&= -\tfrac{3}{2} (D^* F)_{r\alpha}^\beta - \left(\tfrac{x}{2} \lrcorner DD^* F \right)_{r\alpha}^\beta + \tfrac{x_p}{2} \nabla_r \nabla_i F_{ip\alpha}^\beta + \left([F, D^* F]^\# \right)_{r\alpha}^\beta.
\end{aligned}$$

We simplify the third term, nothing vanishing due to the product of skew and symmetric matrices

$$\begin{aligned}
\tfrac{x_p}{2} \nabla_r \nabla_i F_{ip\alpha}^\beta &= -x_p \nabla_r \left(\tfrac{x_s}{2} F_{sp\alpha}^\beta \right) \\
&= \left(-\tfrac{x_p}{2} \delta_{rs} F_{sp\alpha}^\beta - x_s x_p \nabla_r F_{sp\alpha}^\beta \right) \\
&= -x_p F_{rp\alpha}^\beta \\
&= \tfrac{1}{2} (D^* F)_{r\alpha}^\beta.
\end{aligned}$$

Applying this to the above computation we conclude that

$$(D^* D(D^* F))_{r\alpha}^\beta = -(D^* F)_{r\alpha}^\beta - \left(\tfrac{x}{2} \lrcorner DD^* F \right)_{r\alpha}^\beta + \left([F, D^* F]^\# \right)_{r\alpha}^\beta.$$

Then rearranging the equality we have (4.7), as desired. The results follow. \square

Definition 4.4. Given $\nabla \in \mathfrak{S}$ and $\lambda \in \mathbb{R}$, let

$$\chi_\lambda := \{ B \in \Lambda^1(M) : LB = \lambda B \}.$$

Theorem 4.5. Let $\nabla \in \mathfrak{S}$ have polynomial energy growth. Then ∇ is \mathcal{F} -stable if and only if one has the conditions:

- (1) $\chi_{-1} = \{ \rho D^* F : \rho \in \mathbb{R} \},$
- (2) $\chi_{-1/2} = \{ V \lrcorner F : V \in \mathbb{R}^n \},$
- (3) $\chi_\lambda = \{ 0 \}$ for any $\lambda < 0$ and $\lambda \notin \{ -1, -\frac{1}{2} \}.$

Proof. Fix $B \in \Lambda^1(E)$ and decompose it as

$$(4.8) \quad B := \varsigma D^* F + (\varrho \lrcorner F) + \varpi,$$

where $\varsigma \in \mathbb{R}$, $\varrho \in \mathbb{R}^n$, and $\varpi \in \Lambda^1(E)$, and, for all $V \in \mathbb{R}^n$,

$$(4.9) \quad \int_{\mathbb{R}^n} \langle \varpi, D^* F \rangle G dV = \int_{\mathbb{R}^n} \langle \varpi, (V \lrcorner F) \rangle G dV = 0.$$

Using Corollary 3.10 and identities from Corollary 3.4 we have

$$(4.10) \quad \begin{aligned} \left. \frac{d^2}{ds^2} [\mathcal{F}_{0,1}(\nabla_s)] \right|_{s=0} &= -4t^2 \int_{\mathbb{R}^n} |D^* F|^2 G dV - 2 \int_{\mathbb{R}^n} |F \lrcorner \dot{x}|^2 G dV + 4 \int_{\mathbb{R}^n} \langle B, L(B) \rangle G dV \\ &\quad - 4 \int_{\mathbb{R}^n} \langle B, -2tD^* F + \dot{x} \lrcorner F \rangle G dV \\ &= -4t^2 \int_{\mathbb{R}^n} |D^* F|^2 G dV - 2 \int_{\mathbb{R}^n} |F \lrcorner \dot{x}|^2 G dV \\ &\quad + 4 \int_{\mathbb{R}^n} \langle \varsigma D^* F + (\varrho \lrcorner F) + \varpi, -\varsigma D^* F - \frac{1}{2}(\varrho \lrcorner F) + L(\varpi) \rangle G dV \\ &\quad - 4 \int_{\mathbb{R}^n} \langle \varsigma D^* F + (\varrho \lrcorner F) + \varpi, -2tD^* F + \dot{x} \lrcorner F \rangle G dV \\ &= -4t^2 \int_{\mathbb{R}^n} |D^* F|^2 G dV - 2 \int_{\mathbb{R}^n} |F \lrcorner \dot{x}|^2 G dV \\ &\quad - 4\varsigma^2 \int_{\mathbb{R}^n} |D^* F|^2 G dV - 2 \int_{\mathbb{R}^n} |\varrho \lrcorner F|^2 G dV + 4 \int_{\mathbb{R}^n} \langle \varpi, L(\varpi) \rangle dV \\ &\quad + 8\varsigma t \int_{\mathbb{R}^n} |D^* F|^2 G dV - 4 \int_{\mathbb{R}^n} \langle \varrho \lrcorner F, \dot{x} \lrcorner F \rangle G dV \\ &= -4(t - \varsigma)^2 \int_{\mathbb{R}^n} |D^* F|^2 G dV - 2 \int_{\mathbb{R}^n} |F \lrcorner (\varrho + \dot{x})|^2 G dV + 4 \int_{\mathbb{R}^n} \langle \varpi, L(\varpi) \rangle dV. \end{aligned}$$

Choosing $\varsigma = t$ and $\varrho = -\dot{x}$, we have that

$$\left. \frac{d^2}{ds^2} [\mathcal{F}_{0,1}(\nabla_s)] \right|_{s=0} = 4 \int_{\mathbb{R}^n} \langle \varpi, L(\varpi) \rangle dV.$$

Both directions of the theorem follow from this calculation. \square

4.2. Gap theorem. In this subsection we establish Theorem 1.1. To begin we prove a lemma showing that self-shrinking Yang-Mills connections are flat.

Lemma 4.6. *Suppose ∇ is a soliton with bounded curvature satisfying $D_\nabla^* F_\nabla = 0$. Then $F_\nabla = 0$.*

Proof. Suppose to the contrary that ∇ is a nonflat soliton and Yang-Mills connection. As a soliton, by Proposition 2.10, there exists a gauge such that for all $\lambda \in \mathbb{R}$, the connection coefficient matrix satisfies $\Gamma(x, t) = \lambda \Gamma(\lambda x, \lambda^2 t)$. Thus the curvature scales as

$$F_\nabla(x, t) = \lambda^2 F_\nabla(\lambda x, \lambda^2 t),$$

with the given connection ∇ as the time -1 slice. Using this we note that because ∇ is nontrivial there exists some $y \in \mathbb{R}^n$ at which the following limit holds:

$$(4.11) \quad \lim_{t \rightarrow 0} |F_\nabla(y\sqrt{-t}, t)| = \lim_{t \rightarrow 0} \frac{1}{t} |F_\nabla(y, -1)| = \infty.$$

In particular we have $\sup_{x \in \mathbb{R}^n, t \in [-1, 0)} |F_\nabla(x, t)| = \infty$. Simultaneously, since $D_\nabla^* F_\nabla = 0$ and solutions to the Yang-Mills flow on \mathbb{R}^n with bounded curvature are unique, we obtain that $\frac{\partial}{\partial t} \Gamma = 0$ for all $(x, t) \in \mathbb{R}^n \times [-1, 0)$. Therefore we have that for all $x \in \mathbb{R}^n$, then $|F_\nabla(x, t)| =$

$|F_\nabla(x, -1)|$. This implies $\sup_{x \in \mathbb{R}^n, t \in [-1, 0)} |F_\nabla(x, t)| = \sup_{x \in \mathbb{R}^n} |F_\nabla(x, -1)| < \infty$, a contradiction. The result follows. \square

Proof of Theorem 1.1. Since $|F|$ is bounded, it follows from local smoothing estimates for Yang-Mills flow ([20] Theorem 2.2) that $|D^*F|$ and $|\nabla D^*F|$ are also uniformly bounded. Integration by parts yields that

$$\begin{aligned} - \int_{\mathbb{R}^n} |\nabla D^*F|^2 G dV &= \int_{\mathbb{R}^n} (\nabla_i (D^*F)_{\ell\delta}^\beta) (\nabla_i (D^*F)_{\ell\beta}^\delta) G dV \\ &= - \int_{\mathbb{R}^n} (D^*F)_{\ell\delta}^\beta (\nabla_i \nabla_i (D^*F)_{\ell\beta}^\delta) G dV - \int_{\mathbb{R}^n} (D^*F)_{\ell\delta}^\beta (\nabla_i (D^*F)_{\ell\beta}^\delta) \nabla_i G dV \\ &= - \int_{\mathbb{R}^n} (D^*F)_{\ell\delta}^\beta (\Delta (D^*F)_{\ell\beta}^\delta) G dV + \int_{\mathbb{R}^n} (D^*F)_{\ell\delta}^\beta \nabla_i (D^*F)_{\ell\beta}^\delta \frac{x_i}{2} G dV. \end{aligned}$$

Thus from the Bochner formula we have

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla D^*F|^2 G dV &= \int_{\mathbb{R}^n} \langle D^*F, -\Delta D^*F + \frac{x}{2} \lrcorner \nabla D^*F \rangle G dV \\ &= \int_{\mathbb{R}^n} \langle D^*F, \Delta_D D^*F + [F, D^*F]^\# + \frac{x}{2} \lrcorner \nabla D^*F \rangle G dV. \end{aligned}$$

Note by Lemma 2.1 that $D^*D^*F = 0$. Applying the definition of L to the above expression we obtain that, since D^*F is an eigenfunction of L by Lemma 4.3,

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla D^*F|^2 G dV &= \int_{\mathbb{R}^n} \langle D^*F, L(D^*F) + 2[F, D^*F]^\# - (\frac{x}{2} \lrcorner DD^*F) + \frac{x}{2} \lrcorner \nabla D^*F \rangle G dV \\ &= \int_{\mathbb{R}^n} \langle D^*F, -D^*F + 2[F, D^*F]^\# - (\frac{x}{2} \lrcorner DD^*F) + \frac{x}{2} \lrcorner \nabla D^*F \rangle G dV. \end{aligned}$$

In particular, we focus on simplifying the expression $(x \lrcorner \nabla D^*F - x \lrcorner DD^*F)$. This can be simplified by introducing a divergence term and applying the first Bianchi identity, and Lemma 2.1 once more,

$$\begin{aligned} (x \lrcorner \nabla D^*F - x \lrcorner DD^*F)_{m\alpha}^\beta &= x_i \nabla_i (D^*F)_{m\alpha}^\beta - x_i D_i (D^*F)_{m\alpha}^\beta \\ &= x_i \nabla_m (D^*F)_{i\alpha}^\beta \\ &= -x_i \nabla_m \nabla^k F_{ki\alpha}^\beta \\ &= -\nabla_m (x_i \nabla^k F_{ki\alpha}^\beta) + \nabla^k F_{km\alpha}^\beta \\ &= -\nabla_m [\nabla^k (x_i F_{ki\alpha}^\beta) - F_{kk\alpha}^\beta] - (D^*F)_{m\alpha}^\beta \\ &= -\nabla_m \nabla^k (D^*F)_{k\alpha}^\beta - (D^*F)_{m\alpha}^\beta \\ &= -(D^*F)_{m\alpha}^\beta. \end{aligned}$$

Inserting this above and applying the Cauchy Schwartz inequality yields

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla D^*F|^2 G dV &= -\frac{3}{2} \int_{\mathbb{R}^n} |D^*F|^2 G dV + 2 \int_{\mathbb{R}^n} \langle D^*F, [D^*F, F]^\# \rangle G dV \\ &\leq (4|F| - \frac{3}{2}) \int_{\mathbb{R}^n} |D^*F|^2 G dV. \end{aligned}$$

Therefore for ∇ with $|F_\nabla| \leq \frac{3}{8}$, we have that $\|(\nabla D^*F)\sqrt{G}\|_{L^2} = 0$, which implies that D^*F is parallel. Since ∇ is a soliton we have $D^*F = \frac{x}{2} \lrcorner F$, then at $x = 0$, D^*F vanishes and thus since D^*F is parallel, $D^*F = 0$ for all x . By Lemma 4.6 it follows that ∇ is flat, as desired. \square

5. ENTROPY STABILITY

In this section we combine the results of the previous sections to establish that for a soliton with polynomial energy growth the entropy is achieved at $(0, 1)$, and moreover it is uniquely achieved at this point unless the connection has flat directions. The strategy is very similar to ([3] Lemma 7.10). This culminates in the proof of Theorem 1.2.

Definition 5.1. We say that a connection ∇ is *cylindrical* if there is a constant vector field V such that

$$V \lrcorner F_\nabla \equiv 0.$$

Definition 5.2. Given a one-parameter family of connections ∇_s , $s \in I$, let

$$\Xi : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \times I : (x, t, s) \mapsto \mathcal{F}_{x,t}(\nabla_s).$$

Where $\Xi(x, t) := \Xi(x, t, 0)$.

Proposition 5.3. *Suppose that $\nabla \in \mathfrak{S}$ is a connection with polynomial energy growth which is not cylindrical. Given $\epsilon > 0$ there exists $\delta > 0$ such that*

$$(5.1) \quad \sup \{ \mathcal{F}_{x_0, t_0}(\nabla) : |x_0| + |\log t_0| > \epsilon \} < \lambda(\nabla) - \delta.$$

Proof. We show that if ∇ is not cylindrical then Ξ has a strict (global) maximum at $(0, 1)$. We do this by showing that $(0, 1)$ is the unique critical point and then showing the second derivative at $(0, 1)$ is strictly negative. First we show that Ξ has a strict local maximum at $(0, 1)$, then we show that Ξ decreases along a family of paths through the space-time domain emanating from $(0, 1)$ whose union is the entire domain.

For the first step, since ∇ is a soliton then by Proposition 3.6 the gradient of Ξ vanishes at the point $(0, 1)$, which is therefore a critical point. The second variation formula for \mathcal{F}_{x_0, t_0} computed in Proposition 3.9 applied to a fixed ∇ and evaluated along a path $(sy, 1 + sh)$ for $s > 0$ and $h \in \mathbb{R}$ yields that

$$(5.2) \quad \frac{\partial^2}{\partial s^2} (\Xi(sy, 1 + sh)) = -2(1 + sh) \left(2h^2 \int_{\mathbb{R}^n} |D^* F|^2 G_s dV + \int_{\mathbb{R}^n} |y \lrcorner F|^2 G_s dV \right).$$

Note that (5.2) is nonpositive provided $(1 + sh) \geq 0$. The first term vanishes only if $h = 0$ or when ∇ is a Yang-Mills connection, and therefore flat by Proposition 3.5, but we assume ∇ is nonflat. Meanwhile, the second term vanishes only when $y \lrcorner F = 0$, which is not allowed since we assume ∇ is not cylindrical. We thus conclude that Ξ has a strict local maximum at $(0, 1)$.

We next show that for a given $y \in \mathbb{R}^n$ and $a \in \mathbb{R}$, one has $\frac{\partial}{\partial s} (\Xi(sy, 1 + as^2)) \leq 0$ for all $s > 0$ with $1 + as^2 > 0$. We begin with Corollary 3.3, replacing $x_0 \mapsto x_s$ and $t_0 \mapsto t_s$ and $G_0 \mapsto G_s$. We differentiate Ξ using the variation formula (3.10) for \mathcal{F}_{x_0, t_0} with $\dot{\Gamma} = 0$ to obtain

$$(5.3) \quad \frac{\partial}{\partial s} (\Xi(x_s, t_s)) = \dot{t}_s \int_{\mathbb{R}^n} \left(t_s \left(\frac{4-n}{2} \right) + \frac{|x - x_s|^2}{4} \right) |F_\nabla|^2 G_s dV + t_s \int_{\mathbb{R}^n} \frac{\langle y, x - x_s \rangle}{2} |F_\nabla|^2 G_s dV.$$

We insert the two identities (a) and (b) of Corollary 3.3 into (5.3):

$$\begin{aligned}
\frac{\partial}{\partial s} (\Xi(x_s, t_s)) &= t_s \int_{\mathbb{R}^n} \left(t_s \left(\frac{4-n}{2} \right) + \frac{|x-x_s|^2}{4} + \frac{\langle y, x-x_s \rangle}{2} \right) G_s |F|^2 dV \\
&= t_s \int_{\mathbb{R}^n} F_{pu\alpha}^\beta F_{iu\beta}^\alpha (x(t_s-1) + x_s)^p (x-x_s)^i G_s dV \\
&\quad + 2t_s \int_{\mathbb{R}^n} F_{pu\alpha}^\beta F_{iu\beta}^\alpha (x(t_s-1) + x_s)^p y^i G_s dV \\
&= \int_{\mathbb{R}^n} F_{pu\alpha}^\beta F_{iu\beta}^\alpha (x(t_s-1) + x_s)^p (t_s(x-x_s)^i + 2t_s y^i) G_s dV.
\end{aligned}$$

We then evaluate Ξ at $x_s = sy$ and $t_s = 1 + as^2$ to obtain

$$\begin{aligned}
\frac{\partial}{\partial s} (\Xi(sy, 1 + as^2)) &= \int_{\mathbb{R}^n} F_{pu\alpha}^\beta F_{iu\beta}^\alpha (as^2 x + sy)^p (2as(x-sy)^i + 2(1+as^2)y^i) G_s dV \\
&= -2s \int_{\mathbb{R}^n} |(asx + y) \lrcorner F|^2 G_s dV.
\end{aligned}$$

Thus the derivative of Ξ is nonpositive over the union of all paths parametrized by $(sy, 1 + ys^2)$. Since these paths union to the entire space-time domain, we conclude the result. \square

Lemma 5.4. *Let $f \in C^\infty(\mathbb{R}^n)$ be some function pointwise bounded above by a polynomial p . Then for all $x_0 \in \mathbb{R}^n$,*

$$\lim_{t_0 \rightarrow 0} \int_{\mathbb{R}^n} f G_0 dV = f(x_0).$$

In particular we have that for a connection ∇ on \mathbb{R}^n with polynomial curvature growth we have that for all $x_0 \in \mathbb{R}^n$,

$$\lim_{t_0 \rightarrow 0} \mathcal{F}_{x_0, t_0}(\nabla) = 0.$$

Proof. The first line is a well-known property of the heat kernel. Since $|F_\nabla|^2$ has polynomial growth, we have

$$\lim_{t_0 \rightarrow 0} \mathcal{F}_{x_0, t_0}(\nabla) = \lim_{t_0 \rightarrow 0} t_0^2 \int_{\mathbb{R}^n} |F_\nabla|^2 G_0 = \left(\lim_{t_0 \rightarrow 0} t_0^2 \right) \left(\lim_{t_0 \rightarrow 0} \int_{\mathbb{R}^n} |F_\nabla|^2 G_0 \right) = 0$$

\square

Proof of Theorem 1.2. If ∇ is not \mathcal{F} -stable then there is a variation ∇_s for $s \in [-2\epsilon, 2\epsilon]$ where $\nabla_0 = \nabla$ which satisfies the following properties:

- (V1) For each variation ∇_s of ∇ , the support of $\nabla_s - \nabla$ is compact.
- (V2) For any paths (x_s, t_s) with $x_0 = 0$ and $t_0 = 1$ we have

$$(5.4) \quad \left. \frac{\partial^2}{\partial s^2} (\mathcal{F}_{x_s, t_s}(\nabla_s)) \right|_{s=0} < 0.$$

For the family ∇_s , let Ξ be as in Definition 5.2. Also, set

$$(5.5) \quad B^\circ(r) := \{(x, t, s) : 0 < |x| + |\log t| + s < r\}.$$

With this definition we claim that there exists $\epsilon' > 0$ so that for $s \neq 0$ and $|s| \leq \epsilon'$ one has

$$(5.6) \quad \lambda(\nabla_s) := \sup_{x_0, t_0} \Xi(x_0, t_0, s) < \Xi(0, 1, 0) = \lambda(\nabla).$$

Following [3] we proceed in five steps:

- (1) Ξ has a strict local maximum at $(0, 1, 0)$.

- (2) $\Xi(\cdot, \cdot, 0)$ has a strict global maximum at $(0, 1, 0)$.
- (3) $\frac{\partial}{\partial s}(\Xi(x_0, t_0, s))$ is uniformly bounded on compact sets.
- (4) For $|x_0|$ sufficiently large, $\Xi(x_0, t_0, s) < \Xi(0, 1, 0)$.
- (5) For $|\log t_0|$ sufficiently large, $\Xi(x_0, t_0, s) < \Xi(0, 1, 0)$.

Together these five pieces will yield the result as detailed at the end of the proof.

Proof of (1): Since ∇ is a soliton, by Corollary 3.7, given a path (x_s, t_s) with $(x_0, t_0) = (0, 1)$ and a variation ∇_s of ∇ , we have $\frac{\partial}{\partial s}(\Xi(x_s, t_s, s))|_{s=0} = 0$, which implies that $(0, 1, 0)$ is a critical point of Ξ . Consider one such path of the form $(sy, 1 + as)$ for $y \in \mathbb{R}^n$, $a \in \mathbb{R}$ and some variation of ∇ given by ∇_{bs} for some $b \neq 0$. Then we have that, by property (V2),

$$(5.7) \quad \frac{\partial^2}{\partial s^2}(\Xi(sy, 1 + as, bs)) \Big|_{s=0} = b^2 \frac{\partial^2}{\partial s^2}(\mathcal{F}_{x_s, y_s}(\nabla_s)) \Big|_{s=0} \leq 0,$$

where here $x_s = sy$ and $t_s = 1 + as$.

Now we consider the second variation when $b = 0$. As an immediate application of Proposition 5.3 we have that $\frac{\partial^2}{\partial s^2}(\Xi(sy, 1 + as, 0))|_{s=0} < 0$. Therefore $\nabla^2 \Xi$, the Hessian of Ξ , is negative definite at $(0, 1, 0)$, and thus Ξ attains a strict local maximum at this point. We may choose $\epsilon' \in (0, \epsilon)$ such that for $(x_0, t_0, s) \in B^\circ(\epsilon')$ we have that

$$\Xi(x_0, t_0, s) < \Xi(0, 1).$$

Proof of (2): This is an immediate result of Proposition 5.3. Therefore, $\lambda(\nabla) = \Xi(0, 1)$ and we may choose $\delta > 0$ so that for all points of the form $(x_0, t_0, 0)$ outside $B^\circ(\epsilon'/4)$ we have that

$$\Xi(x_0, t_0) < \Xi(0, 1) - \delta.$$

Proof of (3): Using Proposition 3.6, we see that

$$\frac{\partial}{\partial s}(\Xi(x_0, t_0, s)) = 4t_0^2 \int_{\mathbb{R}^n} \left\langle \dot{\Gamma}_s, D_s^* F_s + \left(\frac{(x - x_0)}{2t_0} \lrcorner F_s \right) \right\rangle G_0 dV.$$

Observe that $\frac{\partial \Xi}{\partial s}$ is continuous in all three variables x_0 , t_0 and s and thus uniformly bounded on compact sets.

Proof of (4): By hypothesis, we may choose $R > 0$ so that the support of $\nabla - \nabla_s$ is contained in $B(R) \subset \mathbb{R}^n$. Let $\rho > 0$ and consider $|x_0| > \rho + R$. Then we have that

$$\begin{aligned} \Xi(x_0, t_0, s) &= t_0^2 \int_{\mathbb{R}^n} |F_{\nabla_s}|^2 G_0 dV \\ &= t_0^2 \int_{B(R)} |F_{\nabla_s}|^2 G_0 dV + t_0^2 \int_{\mathbb{R}^n \setminus B(R)} |F_{\nabla}|^2 G_0 dV \\ &\leq t_0^{\frac{4-n}{2}} (4\pi)^{-\frac{n}{2}} \int_{B(R)} |F_{\nabla_s}|^2 e^{-\frac{|x-x_0|^2}{4t_0}} dV + \Xi(x_0, t_0, 0) \\ &\leq t_0^{\frac{4-n}{2}} (4\pi)^{-\frac{n}{2}} e^{-\frac{\rho^2}{4t_0}} \int_{B(R)} |F_{\nabla_s}|^2 dV + \Xi(x_0, t_0, 0). \end{aligned}$$

By compactness of the domain $B(R) \times [-2\epsilon, 2\epsilon]$ we know that $\int_{B(R)} |F_{\nabla_s}|^2 dV < C_R$ for some $C_R \in \mathbb{R}$. Therefore we conclude that

$$(5.8) \quad \Xi(x_0, t_0, s) \leq (4\pi)^{-\frac{n}{2}} C_R t_0^{\frac{4-n}{2}} e^{-\frac{\rho^2}{4t_0}} + \Xi(x_0, t_0, 0).$$

Define the quantity

$$(5.9) \quad \mu_\rho(\tau) := \tau^{\frac{4-n}{2}} e^{-\frac{\rho^2}{4\tau}}.$$

We note in particular that

$$\mu_1\left(\frac{\tau}{\rho^2}\right) = \left(\frac{\tau}{\rho^2}\right)^{\frac{4-n}{2}} e^{-\frac{\rho^2}{4\tau}} = \rho^{n-4} \mu_\rho(\tau).$$

The function μ_1 is clearly continuous and therefore bounded and also satisfies the following limit for $\alpha \in \{\infty, 0\}$,

$$\lim_{\tau \rightarrow \alpha} \mu_1(\tau) = \lim_{\tau \rightarrow \alpha} \tau^{\frac{4-n}{2}} e^{-\frac{1}{4\tau}} = 0.$$

We thus conclude that

$$(5.10) \quad \lim_{\rho \rightarrow \infty} \left(\sup_{\tau > 0} \mu_\rho(\tau) \right) = \lim_{\rho \rightarrow \infty} \sup_{\tau > 0} \left(\rho^{4-n} \mu_1\left(\frac{\tau}{\rho^2}\right) \right) = 0.$$

Therefore, as a consequence of (2) combined with this above limit, we conclude that for $|x_0|$ sufficiently large we have that $\Xi(x_0, t_0, s) < \Xi(0, 1, 0)$, as desired.

Proof of (5): We first perform the following manipulation

$$(5.11) \quad \begin{aligned} \Xi(x_0, t_0, s) &= t_0^2 \int_{\mathbb{R}^n} |F_{\nabla_s}|^2 G_0 dV \\ &= t_0^2 \int_{B(R)} |F_{\nabla_s}|^2 G_0 dV + t_0^2 \int_{\mathbb{R}^n \setminus B(R)} |F_{\nabla_s}|^2 G_0 dV \\ &\leq t_0^{\frac{4-n}{2}} (4\pi)^{-\frac{n}{2}} \int_{B(R)} |F_{\nabla_s}|^2 G_0 dV + \Xi(x_0, t_0, 0) \\ &\leq C_R t_0^{\frac{4-n}{2}} (4\pi)^{-\frac{n}{2}} + \Xi(x_0, t_0, 0). \end{aligned}$$

As a result of this, we also obtain the estimate

$$\sup_{t_0 \geq 1} \Xi(x_0, t_0, s) \leq C_R (4\pi)^{-\frac{n}{2}} + \lambda(\nabla).$$

We break into two cases. First, suppose t_0 is very large. Combining (5.11) with part (2) we obtain the claim. The case when t_0 is small, in particular $t_0 \leq 1$, is more difficult. Appealing to Lemma 3.6 with $t_0 = 1$, we have that for some fixed $R > 0$

$$\begin{aligned} \frac{\partial}{\partial t_0} (\Xi(x_0, t_0, s)) &= \int_{\mathbb{R}^n \setminus B(R)} \left(t_0 \left(\frac{4-n}{2} \right) + \frac{|x - x_0|^2}{4} \right) |F_{\nabla}|^2 G_0 dV \\ &\quad + \int_{B(R)} \left(t_0 \left(\frac{4-n}{2} \right) + \frac{|x - x_0|^2}{4} \right) |F_{\nabla_s}|^2 G_0 dV \\ &\geq t_0 \int_{\mathbb{R}^n} \left(\frac{4-n}{2} \right) |F_{\nabla_s}|^2 G_0 + \left(\frac{4-n}{2} \right) C_R t_0. \end{aligned}$$

Arguing similarly to Lemma 5.4, the integral on the left is bounded. Furthermore since $t_0 \leq 1$ we have that for some $C_0 \in \mathbb{R}$,

$$(5.12) \quad \frac{\partial}{\partial t_0} (\Xi(x_0, t_0, s)) \geq -C_0.$$

Note that C_0 is independent of x_0 , t_0 , and s subject to the restriction $|x_0| < R$. Using Lemma 5.4 and Step (2), we have that

$$(5.13) \quad \Xi(0, 1, 0) = \lambda(\nabla) > 0.$$

Choose $\alpha > 0$ so that $3\alpha < \lambda(\nabla)$, and choose $t_\alpha = \frac{\alpha}{C_0}$. For any $x \in \mathbb{R}^n$ and $s \in [-\epsilon, \epsilon]$, by Lemma 5.4 there exists some $t_{x,s} > 0$ such that for all $t_0 \leq t_{x,s}$ we have $|\Xi(x, t_0, s)| < \alpha$.

On the set $\overline{B(R+1)} \times [-\epsilon, \epsilon]$ we will construct a finite open cover as follows. The cover consists of balls b_i of radius $r_i > 0$ centered at (x_i, t_i) . Each b_i has an associated time $t_i \leq \min\{t_\alpha, 1\}$ where

- (1) Given (x, s) there exists an index $i(x, s)$ such that $(x, s) \in b_{i(x, s)}$.
- (2) For each b_i the associated t_i is such that

$$\Xi(x, t_i, s)|_{b_i} < \alpha.$$

Note that this choice follows from the existence of $t_{x, s}$ and the continuity of Ξ .

Choosing a finite subcover of the b_i 's we let \bar{t} be the minimum of all corresponding t_i . Then as a result of the derivative (5.12) we have that for any triple (x, t_0, s) with $s \in [-\epsilon, \epsilon]$, and $x \in \overline{B_R}$, and $t_0 \leq \bar{t}$,

$$\Xi(x, t_0, s) \leq \Xi(x, t_{i(x, s)}, s) + C_0 (t_{i(x, s)} - t_0) \leq 2\alpha < \lambda(\nabla).$$

Claim (5) follows.

Given claims (1)-(5) we finish the proof by dividing the domain into regions corresponding to the size of $|x_0| + |\log t_0|$. Using (1), when s is sufficiently small there exists some $r > 0$ such that $\Xi(x_0, t_0, s) < \Xi(x_0, t_0, 0)$ for (x_0, t_0) within the following region

$$\mathfrak{R}_1 := \{(x_0, t_0) : |x_0| + |\log t_0| < r\}.$$

Using (4) and (5) there exists an $R > 0$ such that $\Xi(x_0, t_0, s) < \Xi(x_0, t_0, 0)$ for (x_0, t_0) in the following region.

$$\mathfrak{R}_2 := \{(x_0, t_0) : |x_0| + |\log t_0| > R\}.$$

Therefore it remains to consider

$$(5.14) \quad \mathfrak{R}_3 := \{(x_0, t_0) : R > |x_0| + |\log t_0| > r\}.$$

Given $(x_0, t_0) \in \mathfrak{R}_3$, we know by (2) that $\Xi(x_0, t_0, 0) < \lambda(\nabla)$, and by (3) that the s derivative of Ξ is uniformly bounded. So we may choose a $\delta > 0$ such that Ξ restricted to the region $\mathfrak{R}_3 \times [-\delta, \delta]$ is bounded above by $\lambda(\nabla)$. Therefore, (5.6) holds on $\bigcup_{i=1}^3 \mathfrak{R}_i$ and as this union constitutes the entire space-time domain, the result follows. \square

6. GASTEL SHRINKERS

In this section we recall Gastel's construction [4] of $SO(n)$ -shrinking solitons, and compute their entropies. Let $\{\zeta_i\}_{i=1}^n \subset \mathfrak{so}(n)$, be the basis given by, for $\alpha, \beta \in [1, n] \cap \mathbb{N}$,

$$\zeta_{i\alpha}^\beta := \delta_i^\beta x_\alpha - \delta_{i\alpha} x^\beta.$$

Now let r denote the radius on \mathbb{R}^n , and fix some function $\eta : [0, \infty) \rightarrow \mathbb{R}$. Consider the $SO(n)$ -equivariant connections ∇ with coefficient matrices given by

$$(6.1) \quad \Gamma_{i\alpha}^\beta(x) := -\frac{\eta(r)}{r^2} \zeta_{i\alpha}^\beta(x).$$

Proposition 6.1. *For $5 \leq n \leq 9$, and*

$$(6.2) \quad \eta(r) := \frac{r^2}{a_n r^2 + b_n},$$

where

$$(6.3) \quad a_n := \sqrt{\frac{n-2}{8}}, \quad b_n = 3(n-2) - \frac{1}{\sqrt{2}}(n+2)(n-2)^{1/2} \geq 0,$$

the connection ∇ defined by 6.1 is a shrinking soliton.

Proof. As computed in [4] §2.1, under the ansatz of (6.1), the Yang-Mills flow reduces to

$$(6.4) \quad \eta_t = \eta_{rr} + (n-3)\frac{\eta_r}{r} - (n-2)\frac{\eta(\eta-1)(\eta-2)}{r^2}.$$

With $\eta(r)$ as in (6.2) and a_n, b_n as in (6.3), we set

$$\eta(r, t) = \eta\left(\frac{r}{\sqrt{-t}}\right).$$

We now compute various derivatives.

$$(6.5) \quad \begin{aligned} \eta(r, t) &= \frac{r^2}{a_n r^2 - b_n t} \\ \eta_t(r, t) &= \frac{b_n r^2 t}{(a_n r^2 - b_n t)^2} \\ \eta_r(r, t) &= \frac{-2r b_n t}{(a_n r^2 - b_n t)^2}, \\ \eta_{rr}(r, t) &= \frac{(6a_n b_n r^2 t + 2b_n^2 t^2)}{(a_n r^2 - b_n t)^3} \\ (\eta - 1) &= \frac{(1 - a_n)r^2 + b_n t}{(a_n r^2 - b_n t)}, \\ (\eta - 2) &= \frac{(1 - 2a_n)r^2 + 2b_n t}{(a_n r^2 - b_n t)}. \end{aligned}$$

We plug the identities of (6.5) into (6.4) and obtain

$$\begin{aligned} 0 &= \frac{1}{(a_n r^2 - b_n t)^3} (6a_n b_n r^2 t + 2b_n^2 t^2 + (n-3)(-2a_n b_n t r^2 + 2b_n^2 t^2)) \\ &\quad + \frac{1}{(a_n r^2 - b_n t)^3} (-(n-2)((1-a_n)(1-2a_n)r^4 + (3-4a_n)b_n t r^2 + 2b_n^2 t^2) - a_n b_n r^4 + b_n^2 t r^2). \end{aligned}$$

We collect up the coefficients within the numerator of r to various powers. The coefficient of 1 is given by

$$\begin{aligned} &2b_n^2 t^2 + (n-3)2b_n^2 t^2 - (n-2)2b_n^2 t^2 \\ &= (1+n-3-n+2)2b_n^2 t^2 \\ &= 0. \end{aligned}$$

The coefficient of r^2 is given by

$$\begin{aligned} &6a_n b_n t + (n-3)(-2a_n b_n t) - (n-2)(3-4a_n)b_n t + b_n^2 t \\ &= 2a_n b_n t(n+2) - 3(n-2)b_n t + b_n^2 t. \end{aligned}$$

We will compute and then combine portions of the above quantity. First, consider the product

$$\begin{aligned} a_n b_n &= \left(\frac{n-2}{8}\right)^{1/2} \left(3(n-2) + \frac{1}{\sqrt{2}}(n+2)(n-2)^{1/2}\right) \\ &= \frac{3}{\sqrt{2}}(n-2)^{3/2} + \frac{1}{4}(n-2)(n+2). \end{aligned}$$

Therefore we have

$$(6.6) \quad 2a_n b_n(n+2) = \frac{6}{\sqrt{2}}(n+2)(n-2)^{3/2} - \frac{1}{2}(n-2)(n+2)^2.$$

Additionally there is

$$\begin{aligned}
 (6.7) \quad b_n^2 t &= \left(3(n-2) - \frac{1}{\sqrt{2}}(n+2)(n-2)^{1/2} \right)^2 t \\
 &= 9(n-2)^2 t - \frac{6}{\sqrt{2}}(n+2)(n-2)^{3/2} t + \frac{1}{2}(n+2)^2(n-2)t.
 \end{aligned}$$

Lastly we have

$$(6.8) \quad -3(n-2)b_n t = -9(n-2)^2 t + \frac{1}{2}(n+2)^2(n-2)t.$$

Combining together we have

$$(6.9) \quad (6.6) + (6.7) + (6.8) = 0,$$

as desired. Lastly we consider the coefficient of r^4 .

$$\begin{aligned}
 &-a_n b_n - (n-2)(1-a_n)(1-2a_n) \\
 &= -\frac{3}{2\sqrt{2}}(n-2)^{3/2} + \frac{1}{4}(n-2)(n+2) - (n-2) + \frac{3}{2\sqrt{2}}(n-2)^{3/2} - \frac{(n-2)}{4} \\
 &= \frac{(n-2)}{4} ((n+2) - 4 - (n-2)) \\
 &= 0.
 \end{aligned}$$

Thus we conclude that $\eta(r, t)$ satisfies (6.4) and thus this particular connection is a solution to Yang-Mills flow. Next we verify that $\nabla_{-1}(x)$ is a soliton by verifying the scaling law (2.24) of Lemma 2.12. Observe that

$$\begin{aligned}
 \lambda \Gamma_{i\alpha}^\beta(\lambda x, \lambda^2 t) &= -\lambda \frac{\eta(\lambda r, \lambda^2 t)}{\lambda^2 r^2} \zeta_{i\alpha}^\beta(\lambda x) \\
 &= -\lambda \frac{1}{\lambda^2 r^2} \left(\frac{\lambda^2 r^2}{a_n \lambda^2 r^2 - \lambda^2 b_n t} \right) \left(\delta_i^\beta \lambda x_\alpha - \delta_{i\alpha} \lambda x^\beta \right) \\
 &= -\frac{\eta(r, t)}{r^2} \zeta_{i\alpha}^\beta(x) \\
 &= \Gamma_{i\alpha}^\beta(x, t).
 \end{aligned}$$

We conclude that ∇_{-1} is a soliton. □

Proposition 6.2. *Let ∇_n denote the Gastel soliton on \mathbb{R}^n . Then the entropy of ∇_n is approximately*

n	$\lambda(\nabla_n)$
5	638.121
6	716.109
7	929.899
8	1292.44
9	1865.98

Proof. We note that by Proposition 5.3 it suffices to compute $\mathcal{F}_{0,1}(\nabla)$, which we do numerically. In the midst of the computation to obtain (6.4) within [4] §2.1, if we set $\phi = \frac{-\eta}{r^2}$, then

$$(6.10) \quad F_{jk\alpha}^\beta = (2\phi + r^2\phi^2) \left(\delta_k^\beta \delta_{\alpha j} - \delta_{k\alpha} \delta_j^\beta \right) + \left(\frac{\phi_r}{r} - \phi^2 \right) \left(\delta_k^\beta x_\alpha x_j + \delta_{j\alpha} x^\beta x_k - \delta_{k\alpha} x^\beta x_j - \delta_j^\beta x_\alpha x_k \right).$$

We compute \mathcal{F}_{x_0, t_0} by first considering

$$\begin{aligned}
|F|^2 &= -g^{ik}g^{j\ell} \left(F_{ij\alpha}^\beta \right) \left(F_{k\ell\beta}^\alpha \right) \\
&= -g^{ik}g^{j\ell} \left((2\phi + r^2\phi^2) \left(\delta_j^\beta \delta_{\alpha i} - \delta_{j\alpha} \delta_i^\beta \right) + \left(\frac{\phi_r}{r} - \phi^2 \right) \left(\delta_j^\beta x_\alpha x_i + \delta_{i\alpha} x^\beta x_j - \delta_{j\alpha} x^\beta x_i - \delta_i^\beta x_\alpha x_j \right) \right) \\
&\quad \times \left((2\phi + r^2\phi^2) \left(\delta_\ell^\alpha \delta_{\beta k} - \delta_{\ell\beta} \delta_k^\alpha \right) + \left(\frac{\phi_r}{r} - \phi^2 \right) \left(\delta_\ell^\alpha x_\beta x_k + \delta_{k\beta} x^\alpha x_\ell - \delta_{\ell\beta} x^\alpha x_k - \delta_k^\alpha x_\beta x_\ell \right) \right) \\
&= -g^{ik}g^{j\ell} (2\phi + r^2\phi^2)^2 \left[\left(\delta_j^\beta \delta_{\alpha i} - \delta_{j\alpha} \delta_i^\beta \right) \left(\delta_\ell^\alpha \delta_{\beta k} - \delta_{\ell\beta} \delta_k^\alpha \right) \right]_{T_1} \\
&\quad - g^{ik}g^{j\ell} (2\phi + r^2\phi^2) \left(\frac{\phi_r}{r} - \phi^2 \right) \left[\left(\delta_j^\beta \delta_{\alpha i} - \delta_{j\alpha} \delta_i^\beta \right) \left(\delta_\ell^\alpha x_\beta x_k + \delta_{k\beta} x^\alpha x_\ell - \delta_{\ell\beta} x^\alpha x_k - \delta_k^\alpha x_\beta x_\ell \right) \right]_{T_2} \\
&\quad - g^{ik}g^{j\ell} \left(\frac{\phi_r}{r} - \phi^2 \right) (2\phi + r^2\phi^2) \left[\left(\delta_j^\beta x_\alpha x_i + \delta_{i\alpha} x^\beta x_j - \delta_{j\alpha} x^\beta x_i - \delta_i^\beta x_\alpha x_j \right) \left(\delta_\ell^\alpha \delta_{\beta k} - \delta_{\ell\beta} \delta_k^\alpha \right) \right]_{T_3} \\
&\quad - g^{ik}g^{j\ell} \left(\frac{\phi_r}{r} - \phi^2 \right)^2 \left[\left(\delta_\ell^\alpha x_\beta x_k + \delta_{k\beta} x^\alpha x_\ell - \delta_{\ell\beta} x^\alpha x_k - \delta_k^\alpha x_\beta x_\ell \right) \left(\delta_j^\beta x_\alpha x_i + \delta_{i\alpha} x^\beta x_j - \delta_{j\alpha} x^\beta x_i - \delta_i^\beta x_\alpha x_j \right) \right]_{T_4}.
\end{aligned}$$

We first expand T_1 .

$$\begin{aligned}
T_1 &= \left(\delta_j^\beta \delta_{\alpha i} \delta_\ell^\alpha \delta_{\beta k} - \delta_j^\beta \delta_{\alpha i} \delta_{\ell\beta} \delta_k^\alpha - \delta_{j\alpha} \delta_i^\beta \delta_\ell^\alpha \delta_{\beta k} + \delta_{j\alpha} \delta_i^\beta \delta_{\ell\beta} \delta_k^\alpha \right) \\
&= (\delta_{jk} \delta_{\ell i} - \delta_{j\ell} \delta_{ik} - \delta_{j\ell} \delta_{ik} + \delta_{jk} \delta_{i\ell}) \\
&= 2(\delta_{jk} \delta_{\ell i} - \delta_{j\ell} \delta_{ik}).
\end{aligned}$$

Contracting via multiplication by $g^{ik}g^{j\ell}$ yields

$$\begin{aligned}
g^{ik}g^{j\ell}T_1 &= 2(\delta_{jk}^2 - \delta_{jj}\delta_{kk}) \\
&= -2n(n-1).
\end{aligned}$$

Next we expand T_2 .

$$\begin{aligned}
T_2 &= \delta_j^\beta \delta_{\alpha i} (\delta_\ell^\alpha x_\beta x_k + \delta_{k\beta} x^\alpha x_\ell - \delta_{\ell\beta} x^\alpha x_k - \delta_k^\alpha x_\beta x_\ell) \\
&\quad - \delta_{j\alpha} \delta_i^\beta (\delta_\ell^\alpha x_\beta x_k + \delta_{k\beta} x^\alpha x_\ell - \delta_{\ell\beta} x^\alpha x_k - \delta_k^\alpha x_\beta x_\ell) \\
&= \delta_{i\ell} x_j x_k + \delta_{jk} x_i x_\ell - \delta_{j\ell} x_i x_k - \delta_{ik} x_j x_\ell - \delta_{j\ell} x_i x_k - \delta_{ik} x_j x_\ell + \delta_{i\ell} x_j x_k + \delta_{jk} x_i x_\ell \\
&= 2(\delta_{i\ell} x_j x_k + \delta_{jk} x_i x_\ell - \delta_{j\ell} x_i x_k - \delta_{ik} x_j x_\ell).
\end{aligned}$$

Contracting via multiplication by $g^{ik}g^{j\ell}$ yields

$$\begin{aligned}
g^{ik}g^{j\ell}T_2 &= 2(|x|^2 + |x|^2 - n|x|^2 - n|x|^2) \\
&= -4(n-1)|x|^2.
\end{aligned}$$

Expanding T_3 we obtain

$$\begin{aligned}
T_3 &= \left(\delta_j^\beta x_\alpha x_i + \delta_{i\alpha} x^\beta x_j - \delta_{j\alpha} x^\beta x_i - \delta_i^\beta x_\alpha x_j \right) \delta_\ell^\alpha \delta_{\beta k} \\
&\quad - \left(\delta_j^\beta x_\alpha x_i + \delta_{i\alpha} x^\beta x_j - \delta_{j\alpha} x^\beta x_i - \delta_i^\beta x_\alpha x_j \right) \delta_{\ell\beta} \delta_k^\alpha \\
&= \left(\delta_j^\beta \delta_\ell^\alpha \delta_{\beta k} x_\alpha x_i + \delta_{i\alpha} \delta_\ell^\alpha \delta_{\beta k} x^\beta x_j - \delta_{j\alpha} \delta_\ell^\alpha \delta_{\beta k} x^\beta x_i - \delta_i^\beta \delta_\ell^\alpha \delta_{\beta k} x_\alpha x_j \right) \\
&\quad - \left(\delta_j^\beta \delta_{\ell\beta} \delta_k^\alpha x_\alpha x_i + \delta_{i\alpha} \delta_{\ell\beta} \delta_k^\alpha x^\beta x_j - \delta_{j\alpha} \delta_{\ell\beta} \delta_k^\alpha x^\beta x_i - \delta_i^\beta \delta_{\ell\beta} \delta_k^\alpha x_\alpha x_j \right) \\
&= 2(\delta_{jk} x_\ell x_i + \delta_{i\ell} x_k x_j - \delta_{j\ell} x_k x_i - \delta_{ik} x_\ell x_j).
\end{aligned}$$

Contracting by multiplying $g^{ik}g^{j\ell}$ we obtain

$$g^{ik}g^{j\ell}T_3 = -4(n-1)|x|^2.$$

Lastly we expand T_4 and obtain

$$\begin{aligned} T_4 &= \delta_\ell^\alpha x_\beta x_k \left(\delta_j^\beta x_\alpha x_i + \delta_{i\alpha} x^\beta x_j - \delta_{j\alpha} x^\beta x_i - \delta_i^\beta x_\alpha x_j \right) \\ &\quad + \delta_{k\beta} x^\alpha x_\ell \left(\delta_j^\beta x_\alpha x_i + \delta_{i\alpha} x^\beta x_j - \delta_{j\alpha} x^\beta x_i - \delta_i^\beta x_\alpha x_j \right) \\ &\quad - \delta_{\ell\beta} x^\alpha x_k \left(\delta_j^\beta x_\alpha x_i + \delta_{i\alpha} x^\beta x_j - \delta_{j\alpha} x^\beta x_i - \delta_i^\beta x_\alpha x_j \right) \\ &\quad - \delta_k^\alpha x_\beta x_\ell \left(\delta_j^\beta x_\alpha x_i + \delta_{i\alpha} x^\beta x_j - \delta_{j\alpha} x^\beta x_i - \delta_i^\beta x_\alpha x_j \right) \\ &= (x_j x_k x_\ell x_i + \delta_{i\ell} x_k x_j |x|^2 - \delta_{j\ell} x_k x_i |x|^2 - x_i x_k x_\ell x_j) \\ &\quad + (\delta_{jk} x_\ell x_i |x|^2 + x_i x_\ell x_k x_j - x_j x_\ell x_k x_i - \delta_{ik} x_\ell x_j |x|^2) \\ &\quad - (\delta_{j\ell} x_k x_i |x|^2 + x_i x_k x_\ell x_j - x_j x_k x_\ell x_i - \delta_{i\ell} x_k x_j |x|^2) \\ &\quad - (x_j x_\ell x_k x_i + \delta_{ik} x_\ell x_j |x|^2 - \delta_{jk} x_\ell x_i |x|^2 - x_i x_\ell x_k x_j) \\ &= 2(\delta_{i\ell} x_k x_j |x|^2 - \delta_{j\ell} x_k x_i |x|^2 + \delta_{jk} x_\ell x_i |x|^2 - \delta_{ik} x_\ell x_j |x|^2). \end{aligned}$$

Contracting indices $g^{ik}g^{j\ell}$ we obtain

$$g^{ik}g^{j\ell}T_4 = -4(n-1)|x|^4.$$

Therefore, we conclude that

(6.11)

$$|F|^2 = 2n(n-1)(2\phi + r^2\phi^2)^2 + 8(n-1)r^2(2\phi + r^2\phi^2)\left(\frac{\phi_r}{r} - \phi^2\right) + 4(n-1)r^4\left(\frac{\phi_r}{r} - \phi^2\right)^2.$$

We substitute in $\phi(r) = \frac{-\eta(r)}{r^2}$. Recall that $\phi_r = \partial_r\left(\frac{\eta}{r^2}\right) = \frac{-4(n-1)}{r^4}(r\eta_r - 2\eta - \eta^2)$. Therefore we conclude that

$$\begin{aligned} |F|^2 &= \frac{2n(n-1)}{r^4}(2\eta + \eta^2)^2 + \frac{8(n-1)(2\eta + \eta^2)}{r^4}(r\eta_r - 2\eta - \eta^2) + \frac{4(n-1)}{r^4}(r\eta_r - 2\eta - \eta^2)^2 \\ &= \frac{2(n-1)}{r^4}(n(4\eta^2 + 4\eta^3 + \eta^4) + 4(2r\eta\eta_r + r\eta^2\eta_r - 4\eta^2 - 4\eta^3 - \eta^4)) \\ &\quad + \frac{2(n-1)}{r^4}(2(r^2\eta_r^2 - 4r\eta\eta_r - 2r\eta^2\eta_r + 4\eta^2 + 4\eta^3 + \eta^4)) \\ &= \frac{2(n-1)}{r^4}(\eta^2(4n - 16 + 8) + \eta^3(4n - 16 + 8) + \eta^4(n - 4 + 2) + (8 - 8)r\eta\eta_r + 2r^2\eta_r^2 + (4 - 4)r\eta^2\eta_r) \\ &= \frac{2(n-1)}{r^4}(4\eta^2(n-2) + 4\eta^3(n-2) + \eta^4(n-2) + 2r^2\eta_r^2). \end{aligned}$$

Using the definition of η from (6.2) and incorporating the corresponding identities computed in (6.5) we have

$$\begin{aligned} |F|^2 &= \frac{2(n-1)}{r^4(a_nr^2 + b_n)^4}(4r^4(a_nr^2 + b_n)^2(n-2) + 4r^6(a_nr^2 + b_n)(n-2) + r^8(n-2) + 8r^4b_n^2) \\ &= \frac{2(n-1)(n-2)}{(a_nr^2 + b_n)^4}\left(4(a_n^2r^4 + a_nb_nr^2 + b_n^2) + 4(a_nr^4 + b_nr^2) + r^8 + \frac{8r^4b_n^2}{(n-2)}\right) \\ &= \frac{8(n-1)(n-2)}{(a_nr^2 + b_n)^4}\left[b_n^2 + r^2(a_nb_n + b_n) + r^4\left(a_n^2 + a_n + \frac{2b_n^2}{(n-2)}\right) + \frac{1}{4}r^8\right]. \end{aligned}$$

Using this in the definition of $\mathcal{F}_{0,1}$ and integrating numerically yields the result. \square

REFERENCES

- [1] J. Bourguignon, H. Lawson, *Stability and Isolation Phenomena for Yang-Mills Fields*. Commun. Math. Phys. 79, 189-230 (1981).
- [2] Y. Chen, C. Shen, *Monotonicity formula and small action regularity for Yang-Mills flows in higher dimensions*, Calc. Var. 2 (1994) 389-403.
- [3] T. Colding, W. Minicozzi II, *Generic mean curvature flow I; generic singularities*, Annals vol.175 (2012). Issue 2, 755-833.
- [4] A. Gastel, *Singularities of first kind in the harmonic map and Yang-Mills heat flow*, Math. Z. 242 (2002) 47-62.
- [5] J. Grotowski, *Finite time blow-up for the Yang-Mills heat flow in higher dimensions*, Math Zeit. 237 (2001) 321-333.
- [6] R. Hamilton, *A matrix Harnack estimate for the heat equation*. Comm. Anal. Geom. 1 (1993), no.1, 113-126.
- [7] R. Hamilton, *Monotonicity formulas for parabolic flows on manifolds*. Comm. Anal. Geom. 1 (1993), no.1, 127-137.
- [8] G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Diff. Geom. 31 (1990), no. 1, 285-299.
- [9] T. Ilmanen, *Singularities of mean curvature flow of surfaces*, preprint 1995.
- [10] P. Li, *Cambridge studies in advanced mathematics*, Vol. 134, Cambridge University Press, 2012, New York.
- [11] M. Min-Oo, *An L^2 isolation theorem for Yang-Mills fields* Comp. Math. 47.2 (1982), p.153-163.
- [12] H. Naito, *Finite time blowing-up for the Yang-Mills gradient flow in higher dimensions*, Hokkaido Math. J. 23, no. 3 (1994) 451-464.
- [13] J. Rade, *On the Yang-Mills heat equation in two and three dimensions*. J. Reine Angew. Math. 431 (1992), 123-163.
- [14] A. Schlatter, *Long time behaviour of the Yang-Mills flow in four dimensions*. Anal. Global Anal. Geom. 15, (1997).
- [15] A. Schlatter, M. Struwe, S. Tahvildar-Zadeh, *Global existence of the equivariant Yang-Mills heat flow in four space dimensions*, Amer. J. Math. 120 (1998), no. 1, 117-128.
- [16] M. Struwe, *The Yang-Mills flow in four dimensions*. Calc. Var. Partial Differential Equations. 2 (1994), no. 2 123-150. doi:10.1007/BF01191339.
- [17] M. Struwe, *On the evolution of harmonic maps in higher dimensions*, J. Differential Geom. 28 (1988), no. 3, 485-502.
- [18] M. Taylor, *Partial Differential Equations* Vols. 13, 433. Springer-Verlag, New York, 1996.
- [19] A. Waldron, *Instantons and singularities in the Yang-Mills flow* arXiv:1402.3224
- [20] B. Weinkove, *Singularity Formation in the Yang-Mills Flow*. Calc. Var. Partial Differential Equations 19 (2004), no. 2, 211-220.
- [21] B. White, *The size of the singular set in mean curvature flow of mean-convex sets*, J. Amer. Math. Soc, Vol. 13 (2000), no. 3, 665-695.
- [22] Y. Zhang, *\mathcal{F} -stability of self-similar solutions to harmonic map heat flow*. Calc. Var. (2012), 45:347-366.

E-mail address: clkelleh@uci.edu, jstreets@uci.edu

ROWLAND HALL, UNIVERSITY OF CALIFORNIA, IRVINE, CA 92617